Computing Connectedness: an exercise in computational topology

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September 19, 1997

Abstract

We reformulate the notion of connectedness for compact metric spaces in a manner which may be implemented computationally. In particular, our techniques can distinguish between sets that are connected; have a finite number of connected components; infinitely many connected components; or are totally disconnected. It is hoped that this approach will be useful in studying structures from the phase space of dynamical systems.

AMS classification scheme numbers: XXXXXXX

1 Introduction

The extraction of qualitative features from data is one of the primary goals of experimental science. For data that has a natural geometrical representation, or whose geometry arises from phase space embedding [1, 2, 3], such properties can be obtained by either topological or metrical techniques. Metric structure is often probed by the computation of fractal dimensions [4] or Lyapunov exponents [5]. Topological properties, though more fundamental than metrical ones, are more difficult to extract from data.

Examples of topological techniques include the estimation of Hausdorff dimension using minimal spanning trees [6], the calculation of symbolic dynamics of a flow (for the case when there is an attractor that can be embedded in $\mathbb{R}^3$) by analysis of its "template" [7, 8], and the computation of homology for flows that lie on smooth manifolds [9].

By "computational topology" we mean the study of topological properties of an object which can be computed to some finite accuracy. Often we can improve the accuracy of the computations at the expense of increased computer or experimental effort, and the algorithms we study lead to topology by extrapolation.

This notion can be distinguished from "digital topology" [10] which deals with the topological features of digital images, given by discrete values (typically binary) on a regular grid (typically two-dimensional and square). This important field has many applications including algorithmic pattern recognition which might lead to computer vision systems.

*Useful conversations with R. Easton are gratefully acknowledged. JDM was supported in part by NSF grant number DMS-9623216, VR was supported by XXXX(Liz's grant)

The fundamental concept in this field is that of adjacency, the definition of which depends upon the lattice. Much work in this area has focussed on algorithms for the labeling of components, boundaries [11] and other features of digital images. Basic results include consistent notions for connectedness [10], simple connectedness [12], a digital Jordan curve theorem [13], and algorithms for the Euler characteristic [14, 15] of digital sets.

In this paper we consider notions of connectedness that can be computed by extrapolation from computations with finite precision. We show that Cantor's definition of connectedness leads to an algorithm that is guaranteed to decide if a compact set is connected providing the computations could be continued to arbitrary accuracy. More practically, even when a set can only be represented with finite precision, we can use extrapolation to distinguish between sets that appear to be connected, have a finite number of components, an infinite number of components, or be totally disconnected.

We propose that sets that have an infinite number of components can be characterized by a rate of growth of the number of components with improving resolution, we call this the discreteness index, γ. This rate is analogous to the box counting dimension, and for the simplest such sets, e.g. the middle thirds Cantor set, is numerically equal to the dimension. However, more generally the discreteness index distinguishes between sets with the same dimension.

Similarly, sets with components of zero diameter can be characterized by a disconnectedness index, δ, the rate of decrease of the diameter of the components with resolution.

We give a number of simple examples in this paper. In a paper to follow, we will apply these methods to numerical data. A prime motivation for these algorithms is to study the destruction of invariant tori in maps of the torus or in symplectic maps. It is well known that for circle homeomorphisms or area-preserving twist maps, when an invariant circle is destroyed it becomes an invariant cantor set. The topology of the analogous objects in higher dimensions is not known. Thus numerical studies may help with an understanding of these issues.

2 Connectedness

2.1 Classical Definitions

Connectedness is a very intuitive concept: is the space one contiguous piece or not? The generally accepted definition is that a topological space X is *connected* if and only if it cannot be decomposed into the union of two non-empty, disjoint, closed sets (in the relative topology). If such a decomposition exists then X is said to be *disconnected*– that is, if there are two closed sets U and V such that $U \cap V = \emptyset$ and $U \cup V = X$.

In the case that X is a metric space we say it is *path connected* if, given any $x, y \in X$, there is a continuous function $f : [0,1] \to X$, such that $f(0) = x$ and $f(1) = y$. Path connected spaces are connected, but not necessarily vice versa. For example, the "topologists' sine curve"(figure 5), is the graph of $y = \sin(1/x)$ together with the $[-1,1]$ segment of the y-axis, is connected but not path connected, as it is impossible to get from a point on the curve to a point on the y-axis in finite time.

One object in which we are particularly interested, is the *connected component* of a point. Given $x \in X$ this is the largest connected subset of X containing x. For example, if $X = [0,1] \cup [2,3]$ then the connected component of $\frac{1}{2}$ is $[0,1]$. If the connected component of every point is only the point itself then the set is said to be *totally disconnected*. The rationals are totally disconnected, as is the familiar middle-thirds Cantor set.

2.2 ϵ-Connectedness

Our aim is to reformulate the notion of connectedness in a way that relies on extrapolation, to make it possible to implement a test for this property on a computer. The basic idea is to look at the set with a finite resolution ϵ and see how certain properties change as we let $\epsilon \to 0$.

Given a subset X of a metric space, we say it is *ϵ-disconnected* if it can be decomposed into two sets that are separated by a distance of at least ϵ - i.e. there are two closed subsets, U and V with $U \cup V = X$ and $d(U,V) = \inf_{x\in U, y\in V} d(x,y) > \epsilon$. Otherwise, X is *ϵ-connected.* One would now hope that if X is ϵ-connected for all $\epsilon > 0$, then X is connected in the original sense. This is in fact true, provided that X is also compact, which we demonstrate by proving the contrapositive:

Lemma 1. *If a compact metric space X is disconnected, then it is ϵ-disconnected for some $\epsilon > 0$.*

Proof. Since X is disconnected, there are closed disjoint sets U and V such that $U \cup V = X$. Now suppose $d(U,V) = 0$. This implies that there are sequences $x_n \in U$, $y_n \in V$ with $d(x_n, y_n) \to 0$. Since U and V are compact, there must be convergent subsequences $x_{n_i} \to x^*$ and $y_{n_j} \to y^*$ with $d(x^*, y^*) = 0$. But this implies $x^* = y^*$, which is impossible since U and V are disjoint, so there must be an $\epsilon > 0$ such that $d(U,V) = \epsilon$. □

Our definition is equivalent to an early formulation of connectedness due to Cantor [16] that uses the concept of an *ϵ-chain*: a finite sequence of points $x_0, \ldots, x_n$ with $d(x_i, x_{i+1}) < \epsilon$ for $i = 1, \ldots, n$. Cantor calls a set X connected iff every two points in X can be linked by an ϵ-chain for arbitrarily small ϵ. The equivalence follows since X is ϵ-connected if and only if for all $x, y \in X$, there is an ϵ-chain with $x_0 = x$ and $x_n = y$.

As remarked, these ϵ-definitions agree with the now-standard one only in the special case of compact metric spaces. For instance, the rationals are Cantor-connected but are disconnected in the regular sense. This restriction is not as bad as it might seem, since we are primarily interested in sets that are generated dynamically, such as ω-limit sets of orbits, and these sets are compact when they are bounded.

It is also worth noting that being ϵ-connected for all $\epsilon > 0$ does not imply that X is path connected: cf., the topologists' sine curve again.

We now make a corresponding definition of connected component: S is an *ϵ-component* of X if S is ϵ-connected and $d(X \setminus S, S) > \epsilon$. Given a resolution, ϵ, X has a natural decomposition as the disjoint union of its ϵ-components. The idea behind this paper is that investigating how this decompostion changes as ϵ approaches zero tells us about the connectedness of X. If the only ϵ-component is X itself for all ϵ, then we can use lemma 1 to conclude that X is connected. The theorem below tells us what happens in the more complex case that X is totally disconnected. Let S_ϵ represent an ϵ-component. Recall that the diameter of a set is defined as $\mathrm{diam}(S) = \sup_{x,y\in S} d(x,y)$. We have then:

Theorem 2. *A compact set X is totally disconnected if and only if*

$$\lim_{\epsilon\to 0} \sup_{S_\epsilon} \mathrm{diam}(S_\epsilon) = 0.$$

In order to prove the theorem, we need the following lemma.

Table 1: Summary of the connectedness properties of a set that can be deduced from the limiting behaviour of $C(\epsilon)$ and $D(\epsilon)$.

	$C(\epsilon) \equiv 1$	$\lim_{\epsilon\to 0} C(\epsilon) < \infty$	$\lim_{\epsilon\to 0} C(\epsilon) = \infty$
$\limsup_{\epsilon\to 0} D(\epsilon) = 0$	A single point	A finite, totally disconnected set	An infinite, totally disconnected set
$\limsup_{\epsilon\to 0} D(\epsilon) \neq 0$	A connected set, $D(\epsilon) \equiv \mathrm{diam}(X)$	A finite number of connected components	Infinitely many connected components

Lemma 3. *If X is a compact set, it has a finite number of ϵ-components.*

Proof. Take a covering of X by ϵ-balls. Since X is compact, there is a finite sub-cover. By their definition, each ϵ-component must contain at least one ϵ-ball $\cap X$, so there are a finite number of ϵ-components. $\square$

Proof of the Theorem. By way of obtaining a contradiction to the forward direction, suppose that

$$\lim_{\epsilon\to 0} \sup_{S_\epsilon} \mathrm{diam}(S_\epsilon) = \delta > 0.$$

Take any sequence of $\epsilon_n \to 0$ and construct a tree as follows. At level n list all the ϵ_n-components with diameter $\geq \delta$. There are a finite number of these by the lemma. Order the tree by set inclusion, i.e. an edge connects S_{ϵ_j} and $S_{\epsilon_{j+1}}$ if and only if $S_{\epsilon_{j+1}} \subset S_{\epsilon_j}$. We know that there must be ϵ-components with diameters $\geq \delta$ for all ϵ so this tree must have an infinite branch. This gives a sequence of *nested* components S_{ϵ_n}, with $\mathrm{diam}(S_{\epsilon_n}) \searrow \delta$. Since the sets are nested, they have a limit, $S_* = \lim_{n\to\infty} S_{\epsilon_n} = \bigcap_n S_{\epsilon_n}$. It then follows that S_* is ϵ-connected for all ϵ and $\mathrm{diam}(S_*) = \delta$. This implies S_* has at least two points in it and so X could not be totally disconnected.

The converse follows essentially from the definition. If the diameters of the ϵ-components go to zero then the connected component of any $x \in X$ is just $\{x\}$ so X is totally disconnected. $\square$

Theorem 2 and lemma 3 motivate the introduction of two important quantities that are used to classify the connectedness properties of a set: (1) the number of ϵ-components, $C(\epsilon)$, and (2) the diameters of the components, $D(\epsilon)$. The behaviour of these quantities as $\epsilon \to 0$ tells us whether or not the set is totally disconnected and how many connected components there are. The different cases are summarised in the table. In the next section, we present simple examples that illustrate some of the possible types of set. A general set is more likely to be a union of these basic types.

3 Examples

We focus on two groups of examples: 1-dimensional Cantor sets, and relatives of the Sierpinski gasket. Both were chosen because they are easily described in terms of ϵ-components and therefore good test cases to illustrate the approaches desribed in this paper. Cantor sets have been identified as important structures in dynamics, in particular circle homeomorphisms and area-preserving twist maps. The Sierpinski relatives are 2-dimensional attractors of a family of iterated function systems and are described at length in Peitgen *et al*, [17]. They include a range of topological structures which we might expect to see in invariant sets of higher- dimensional dynamical systems.

In section 3.3 we extract principles from these examples that hold for general iterated function systems.

3.1 Cantor Sets

A Cantor set is a totally disconnected, perfect, compact metric space. Recall that a set is *perfect* if it is equal to the set of its accumulation points. In other words, every point has arbitrarily small neighbourhoods containing infinitely many other points, so no point is isolated. This can be formulated in our ϵ-resolution terms as:

Lemma 4. *A compact set X is perfect if and only if the* $\inf_{S_\epsilon} \operatorname{diam}(S_\epsilon) > 0$ *for all* $\epsilon > 0$.

This should be clear from the definitions.

To simplify notation a little, let $\max D(\epsilon) = \overline{D(\epsilon)}$ and $\min D(\epsilon) = \underline{D(\epsilon)}$.

How do we expect $C(\epsilon)$ and $D(\epsilon)$ to behave as $\epsilon \to 0$? Theorem 2 tells us that $\overline{D(\epsilon)} \to 0$; since a Cantor set is both perfect and totally disconnected, we must have $C(\epsilon) \to \infty$ (see figure 5). A more-accurate description is given by the asymptotic behaviour, which we assume to be a general power law. That is, near $\epsilon = 0$, $\overline{D(\epsilon)} \sim \epsilon^\delta$ and $C(\epsilon) \sim \epsilon^{-\gamma}$. The exponents may be found as the following limits:

$$\begin{aligned} \delta &= \liminf_{\epsilon \to 0} \frac{\log \overline{D(\epsilon)}}{\log \epsilon} \\ \gamma &= \limsup_{\epsilon \to 0} \frac{\log C(\epsilon)}{\log(1/\epsilon)} \end{aligned}$$

We call δ the *disconnectedness* index. The limit is taken to be the lim inf so that we know $\overline{D(\epsilon)} \le D_0 \epsilon^\delta$ for some constant D_0. We think of δ as measuring how sparsely the points are distributed– in fact, it is related to the notion of thickness for 1-dimensional Cantor sets.

The component growth rate, γ, is called the *discreteness* index. It is found as the lim sup to ensure $C(\epsilon) \ge C_0 \epsilon^{-\gamma}$, for some constant C_0.

Middle-α Cantor sets

These Cantor sets arise in piecewise linear horseshoe maps and have zero Lebesgue measure. Let $0 < \alpha < 1$ and consider the Cantor set, $K_\alpha \subset [0,1]$ constructed by successively removing the middle α-proportion of each remaining interval. This construction has a natural correspondence with the ϵ-components. At a given level n, there are $C_n = 2^n$ intervals of equal length $D_n = \frac{1}{2}(1-\alpha)D_{n-1}$, separated by gaps of at least $g_n = \alpha D_{n-1}$. See figure 5. With $D_0 = 1$, the recursion relations may be solved to find $D_n = [\frac{1}{2}(1-\alpha)]^n$ and $g_n = \alpha[\frac{1}{2}(1-\alpha)]^{n-1}$, so that

$$\delta = \lim_{n\to\infty} \frac{\log D_n}{\log g_n} = \lim_{n\to\infty} \frac{n \log[\frac{1}{2}(1-\alpha)]}{(n-1)\log[\frac{1}{2}(1-\alpha)] + \log \alpha} = 1 ,$$

and

$$\gamma = \lim_{n\to\infty} \frac{-\log C_n}{\log g_n} = \lim_{n\to\infty} \frac{-n\log 2}{(n-1)\log[\frac{1}{2}(1-\alpha)] + \log\alpha} = \frac{\log 2}{\log 2 - \log(1-\alpha)} .$$

The disconnectedness index, δ, is independent of α because the Cantor set is constructed in such a way that the ϵ-components and gaps decrease at the same rate. The discreteness index, γ, has the same value as the Hausrdorff dimension.

A Cantor set with positive measure

Now consider a Cantor set with gaps that decrease more rapidly. Let K be the subset of $[0,1]$, obtained by successively removing gaps from the centre of remaining intervals, with widths $g_n = (\frac{1}{2})^{2n-1}(\frac{1}{10})$. The total Lebesgue measure of the gaps is just $\frac{1}{10}$, so the measure of K is $\frac{9}{10}$. After a bit of algebra, we find that for $(\frac{1}{2})^{2n+1}(\frac{1}{10}) \leq \epsilon < (\frac{1}{2})^{2n-1}(\frac{1}{10})$, there are $C_n = 2^n$ components with diameters $D_n = (\frac{1}{2})^n[\frac{9}{10} + (\frac{1}{2})^n(\frac{1}{10})]$. So,

$$\delta = \lim_{n\to\infty} \frac{n\log(\frac{1}{2}) + \log[\frac{9}{10} + (\frac{1}{2})^n(\frac{1}{10})]}{(2n-1)\log(\frac{1}{2}) + \log(\frac{1}{10})} = \frac{1}{2} ,$$

and

$$\gamma = \lim_{n\to\infty} \frac{-n\log 2}{(2n-1)\log(\frac{1}{2}) + \log(\frac{1}{10})} = \frac{1}{2} .$$

Since K has positive Lebesgue measure, its Hausdorff dimension is 1, which is not the same value as γ.

3.2 Relatives of the Sierpinski gasket

The Sierpinski gasket, S, is a fractal generated by the simple iterated function system (IFS): $S = f[S] = f_1[S] \cup f_2[S] \cup f_3[S]$, where, (c.f. figure 5)

$$\begin{aligned} f_1(x,y) &= \tfrac{1}{2}(x,y) \\ f_2(x,y) &= \tfrac{1}{2}(x+1,y) \\ f_3(x,y) &= \tfrac{1}{2}(x,y+1). \end{aligned}$$

Its "relatives" are generated by allowing the similarities to rotate or reflect the image. They all have the same Hausdorff dimension, yet there is a great deal of variation in their topological structure. Peitgen *et al*, [17] identify three different classes: one-dimensional and connected like the gasket; simply connected (i.e. homotopic to a point) or totally disconnected and therefore zero dimensional. There is, in fact, a fourth type which has infinitely many connected components, but a topological dimension of 1. For examples of each type, see figure 5. A shortcoming in our classification scheme is that we cannot distinguish between the different types of connected set: for both the connected cases, we have $C(\epsilon) \equiv 1$ and $D(\epsilon) \equiv 1$. The next two examples are of the disconnected types.

A 2-D Cantor Set

Consider the fractal (figure 5c) generated by:

$$\begin{aligned} f_1(x,y) &= \tfrac{1}{2}(-y+1,x) \\ f_2(x,y) &= \tfrac{1}{2}(y+1,x) \\ f_3(x,y) &= \tfrac{1}{2}(y,-x+2) \end{aligned}$$

The gap sizes are given by $g_n = g_0/2^n$, where g_0 is the largest gap. The diameters of components follow the same pattern, after some initial transient behaviour. Measured in the L_∞ metric: $\overline{D_0} = 1, \overline{D_1} = \overline{D_2} = 1/2, \overline{D_3} = 1/4, \dots, \overline{D_n} = 1/2^{n-1}$. This gives $\delta = 1$, showing that the set is totally disconnected.

The number of components, C_n, is a little harder to determine. Since C_n is just one more than the number of gaps, we do so by first deriving an expression for the latter. Let N_n be the number of gaps of size g_n. Then, with $N_0 = 1$ we have the recursion:

$$N_n = \begin{cases} 3N_{n-1} & \text{if } n \text{ is odd,} \\ 3N_{n-1} - 2\cdot 3^{n/2-1} & \text{if } n \text{ is even.} \end{cases}$$

These are solved to find:

$$N_n = \begin{cases} 2\cdot 3^{n-1} + 3^{(n-1)/2} & \text{if } n \text{ is odd,} \\ 2(3^{n-1} - 3^{n/2-1}) + 3^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

We then compute

$$C_n = 1 + \sum_{j=0}^{n} N_j = \begin{cases} 3^n + 2\cdot 3^{(n-1)/2} & \text{if } n \text{ is odd,} \\ 3^n + 3^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

The leading power is the same for all n, so may use either case to evalute the limit:

$$\gamma = \limsup_{n\to\infty} \frac{\log(C_n)}{\log(1/g_n)} = \lim_{n\to\infty} \frac{\log[3^n + 3^{n/2}]}{\log[2^n/g_0]} = \frac{\log 3}{\log 2}.$$

Since there are no isolated points, the attractor is perfect and therefore a Cantor set as claimed.

Something new

This example is a fractal that illustrates the case from table 2.2 that S has infinitely many connected components yet is not totally disconnected (see figure 5d). It has the same component growth rate as the Cantor set above. The similarities for the IFS are:

$$\begin{aligned} f_1(x,y) &= \tfrac{1}{2}(x,y) \\ f_2(x,y) &= \tfrac{1}{2}(y+1,-x+1) \\ f_3(x,y) &= \tfrac{1}{2}(x,y+1) \end{aligned}$$

Again the gaps decrease simply as $g_n = g_0/2^n$ and the number of components is $C_n = 3^n + 1$, giving $\gamma = \log 3/\log 2$. The ϵ-component diameters, however, do not decrease with resolution. In fact, using the L_∞ metric, the set of diameter values is $D_n = \{1, 1/2, \dots, 1/2^n\}$.

3.3 General Iterated Function Systems

Iterated function systems have the general form: $S = f[S] = f_1[S] \cup f_2[S] \cup \ldots \cup f_m[S]$ where each f_i is a similarity with contraction ratio $0 < r_i < 1$. Since $S = f[S]$, if two connected components of S are separated by a gap of size g_0, then there must be infinitely many separating gaps with sizes $r_1^{s_1} r_2^{s_2} \cdots r_m^{s_m} g_0$ for all integers $s_i \geq 0$. Similarly, if there is an ϵ-connected component of diameter D_0, then there are components (at other resolutions) with diameters $r_1^{s_1} r_2^{s_2} \cdots r_m^{s_m} D_0$. In the case that all the similarity ratios are the same, as with the middle-α Cantor sets and Sierpinski relatives, this simplifies dramatically to $g_n = r^n g_0$ and $D_n = r^n D_0$, where n is a (sufficiently large) integer. Any such IFS with a Cantor set as its attractor must therefore have $\delta = 1$. This is exactly what was found in the examples above. The measure-$\frac{9}{10}$ Cantor set has $\delta = \frac{1}{2}$ because the gap sizes decrease more rapidly than the diameters. This means it cannot be described as the attractor of an IFS consisting of similarities only (although it may be possible to do this if more general functions are allowed).

4 Conclusions

We have reformulated the definition of connectedness for the case of compact metric spaces. By looking at a set with increasingly fine resolution we can determine whether it is connected; has a finite or infinite number of connected components; or is totally disconnected. This is accomplished by determining the limiting behaviour of two readily computable properties– the number of ϵ-components, $C(\epsilon)$, and their diameters, $D(\epsilon)$. Details of the computer implementation of these results will be published in a future paper.

Our approach is more flexible than that of digital topology. Objects are not confined to a regular grid and may be represented by a general point-set of arbitrary dimension. This framework will enable us to investigate the structure of attractors and other important sets in dynamical systems. It will be particularly useful when the phase space has dimension > 3 as it becomes very difficult to visualise such objects.

Other topological properties that would be nice to compute include the notion of "perfect". We have already indicated how the definition can be formulated in our terms. This could be refined further, to some notion of a rate: how fast do points limit on a given point as the resolution is improved? It would also be useful to distinguish between the different "flavours" of connectedness – particularly to tell if the set contained holes.

It should also be easy to adapt our techniques to compute the "thickness" of Cantor sets. This figures prominently in Newhouse's theory of the persistence of homoclinic tangencies.

5 References

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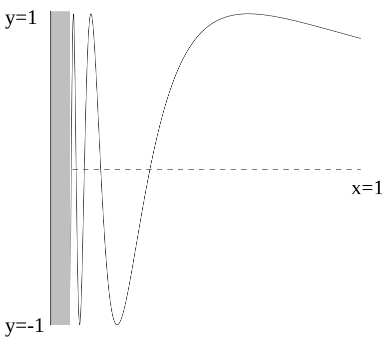


Figure 1: The topologists' sine curve

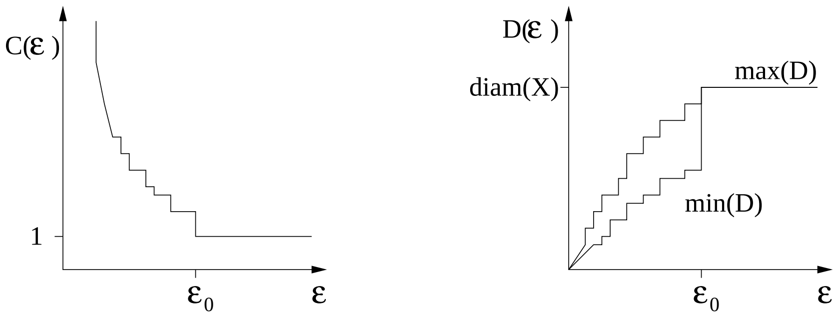


Figure 2: The general behaviour of $C(\epsilon)$ and $D(\epsilon)$ for a Cantor set.

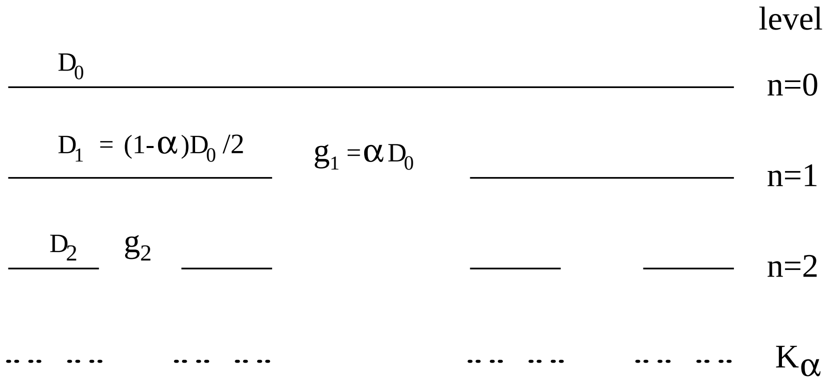


Figure 3: The construction of a K_α Cantor set.

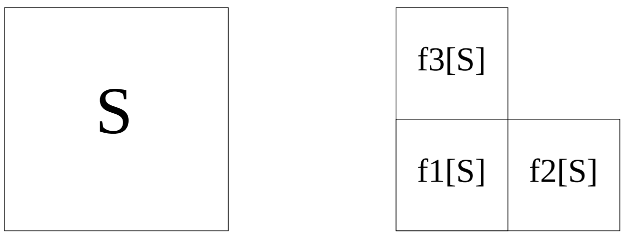
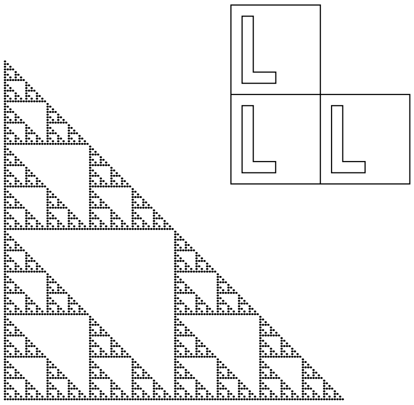


Figure 4: The general template for a Sierpinski relative.



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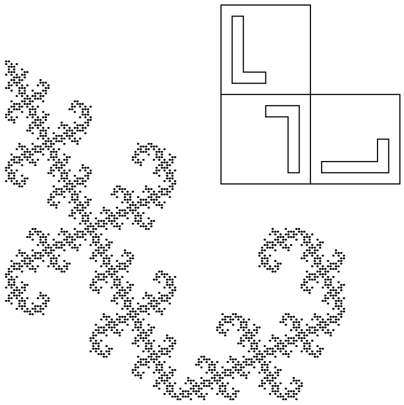

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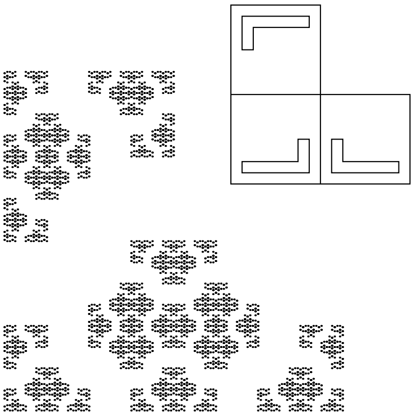
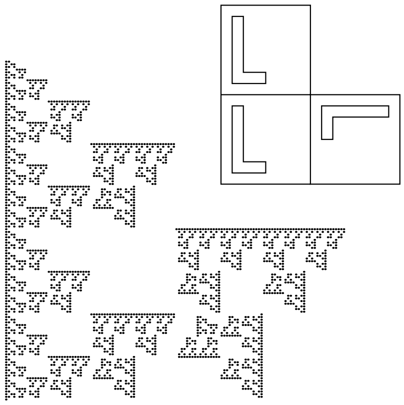


c



d

Figure 5: Four Sierpinski relatives and their templates (the L shows the orientation of the image of S under each similarity.): (a) The Sierpinski gasket; (b) A gasket relative that is simply connected; (c) One that is totally disconnected; (d) A relative with infinitely many connected components;