Stability of Minimal Periodic Orbits

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Abstract

Symplectic twist maps are obtained from a Lagrangian variational principle. It is well known that nondegenerate minima of the action correspond to hyperbolic orbits of the map when the twist is negative definite and the map is two dimensional. We show that for more than two dimensions, periodic orbits with minimal action in symplectic twist maps with negative definite twist are not necessarily hyperbolic. In the proof we show that in the neighborhood of a minimal periodic orbit of period n, the n^{th} iterate of the map is again a twist map. This is true even though in general the composition of twist maps is not a twist map.

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1 Introduction

We consider a discrete Lagrangian system on the configuration space $\mathcal{Q}$, of dimension d. A discrete Lagrangian, $L(x, x')$, $x, x' \in \mathcal{Q}$, is a generating function for a symplectic map $(x', y') = F(x, y)$ on $\mathcal{Q} \times \mathbb{R}^d$, that is implicitly defined by (for a review see [10])

$$y = -L_1(x, x'), \quad y' = L_2(x, x'). \tag{1}$$

The subscripts 1 and 2 denote the derivative with respect to the first or second argument, respectively. We assume that the (local) twist condition, $\det L_{12} \neq 0$, holds, so that x' can be determined, at least locally, as a function of (x, y). The dynamics can also be obtained from a variational principle: define the periodic action by

$$W_{mn} = \sum_{i=0}^{n-1} L(x_i, x_{i+1})\big|_{x_n = x_0 + m} . \tag{2}$$

When the configuration space is the torus, $\mathcal{Q} = \mathbb{T}^d$, we can fix the period of the torus to 1 in every dimension and choose $m \in \mathbb{Z}^d$, otherwise we just set $m = 0$. It is easy to see that every critical point of W_{mn} corresponds to a periodic orbit of F with period n.

A minimal periodic orbit is a nondegenerate, local minimum of W_{mn} (we do not require it to be globally minimizing as in Aubry-Mather theory [10]). When the twist is negative definite, minimal orbits are expected to be important: for example every orbit on an invariant torus (that is a Lagrangian graph) is minimizing [9]. The purpose of this note is to establish the relation – if any – between the fact that the orbit is minimal and its stability type.

Relations between the index of a certain quadratic form (which is not the Hessian of the action) and the stability type of fixed points of symplectic mappings have been obtained in [1]. Similar results specialized to natural maps can be found in [5]. Our approach is different because we specifically look at action minimizing orbits, i.e. the index of the Hessian of the action is 0 (or maximal). In [2] it was shown that there exist Lagrangian *flows* for which the action minimizing equilibrium points or the minimizing periodic orbits are not hyperbolic. This is very similar to our results, but we treat the case of maps with a completely different method. Although it is known that symplectic twist maps have a corresponding time one flow [11], this requires one to treat time dependent Lagrangian flows, which was not done in [2].

Linear stability of a periodic orbit is determined by its multipliers. Let $\{x_1, x_2, \ldots x_n\}$ be a periodic orbit with period n, and let $x_{i+n} = x_i$. The linearization of the map at this orbit gives rise to an eigenvalue problem with eigenvalues that we call μ, the multipliers for the orbit. We define the residue R associated with a multiplier by

$$R = \frac{1}{4}\left(2 - \mu - \frac{1}{\mu}\right) \tag{3}$$

Since the multipliers for a symplectic map come in reciprocal pairs, μ and $1/\mu$, there are d residues in dimension d, and their values completely determine the stability type of the orbit. A multiplier is elliptic, denoted "E" when $\mu = e^{i\phi}$ or equivalently when $0 \leq R \leq 1$. It is inverse hyperbolic, denoted "I", when $1 < R$ and hyperbolic, denoted "H", when $R < 0$. Finally a multiplier is part of a complex quartet when R is complex; we denote this case "CQ". Of course this latter case can occur only when $d \geq 2$.

With the notation

$$D^2 L(x_i, x_{i+1}) = \begin{pmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{B}_i^T & \mathbf{D}_i \end{pmatrix} \tag{4}$$

for the Hessian of L at (x_i, x_{i+1}) we can express the linearized map DF directly in terms of these data as [4]

$$DF(x_i, y_i) = \begin{pmatrix} -\mathbf{B}_i^{-1}\mathbf{A}_i & -\mathbf{B}_i^{-1} \\ \mathbf{B}_i^T - \mathbf{D}_i\mathbf{B}_i^{-1}\mathbf{A}_i & -\mathbf{D}_i\mathbf{B}_i^{-1} \end{pmatrix}. \tag{5}$$

It is often more convenient to obtain the stability of period n orbits directly from the Lagrangian formulation. Using the abbreviation

$$\mathbf{P}_i = \mathbf{A}_i + \mathbf{D}_{i-1}, \tag{6}$$

the linearization of the Euler-Lagrange equations about the orbit is

$$\mathbf{B}_{i-1}^T \delta x_{i-1} + \mathbf{P}_i \delta x_i + \mathbf{B}_i \delta x_{i+1} = 0. \tag{7}$$

The multipliers μ are determined by solving this system subject to the condition that $\delta x_{i+n} = \mu \delta x_i$. This gives the characteristic polynomial $\det \mathbf{M}_n(\mu) = 0$ [9, 7]. Here the matrix $\mathbf{M}$ takes slightly different forms for fixed points and period 2 orbits, and we distinguish these with a subscript that indicates the period:

$$\mathbf{M}_1(\mu) = \mathbf{P}_1 + \mu \mathbf{B}_1 + \frac{1}{\mu}\mathbf{B}_1^T \tag{8}$$

$$\mathbf{M}_2(\mu) = \begin{pmatrix} \mathbf{P}_1 & \mathbf{B}_1 + \mathbf{B}_2^T/\mu \\ \mathbf{B}_1^T + \mu \mathbf{B}_2 & \mathbf{P}_2 \end{pmatrix} \tag{9}$$

$$
\mathbf{M}_n(\mu) = \begin{pmatrix} \mathbf{P}_1 & \mathbf{B}_1 & & & \frac{1}{\mu}\mathbf{B}_n^T \\ \mathbf{B}_1^T & \mathbf{P}_2 & \mathbf{B}_2 & & \\ & & \ddots & & \\ & & \mathbf{B}_{n-2}^T & \mathbf{P}_{n-1} & \mathbf{B}_{n-1} \\ \mu\mathbf{B}_n & & & \mathbf{B}_{n-1}^T & \mathbf{P}_n \end{pmatrix}, \; n > 2\,. \tag{10}
$$

The Hessian of the periodic action W_{mn} is given by $\mathbf{M}_n(1)$. The assumption that the periodic orbits under consideration be minimal therefore is

$$
\mathbf{M}_n(1) > 0. \tag{11}
$$

Note that if a multiplier is on the unit circle $\mu = e^{i\phi}$, then the matrix $\mathbf{M}_n(e^{i\phi})$ is Hermitian.

When $d = 1$, there is a simple relation between the Hessian of the periodic action W_{mn} and the residue [8]:

$$
R = -\frac{1}{4}\frac{\det \mathbf{M}_n(1)}{\prod_{i=1}^{n}(-\mathbf{B}_i)} \tag{12}
$$

where $\mathbf{M}_n(1) = D^2 W_{mn}$ is the Hessian and $\mathbf{B}_i \equiv L_{12}(x_i, x_{i+1})$.

For $d > 1$, there is no such simple relation, though the product of the residues can be written similarly [7]. Equation (12) implies that when $d = 1$ and the twist is negative definite, nondegenerate minimal orbits are hyperbolic. We will show that this is false for $d > 1$: the multipliers of minimal periodic orbits can become elliptic.

In section 2 we analyze minimal fixed points and establish the fact that they can be nonhyperbolic if the twist is either nonsymmetric or indefinite. For 4D maps we completely analyze the structure of minimizing fixed points in the space of three essential parameters. Then our strategy is to reduce the case of minimal periodic orbits to that of a minimal fixed point. Clearly the period n orbits of F are fixed points of the iterated map F^n. However, it is well known [10] that the iterate of a twist map is in general not a twist map. This does not preclude the possibility that the iterated map restricted to the neighborhood of a minimal periodic orbit is a twist map, which we will prove to be the case in section 3. Finally we give two examples of minimizing periodic orbits which are elliptic.

2 Fixed Points

The stability of a fixed point is determined by solutions to $\det \mathbf{M}_1(\mu) = 0$. For a minimal orbit $\mathbf{M}_1(1) = \mathbf{P}_1 + \mathbf{B}_1 + \mathbf{B}_1^T > 0$. Rewriting $\mathbf{M}_1(\mu)$ to isolate this term gives

$$\mathbf{M}_1(\mu) = \mathbf{M}_1(1) + \frac{1}{2}(\mu + \frac{1}{\mu} - 2)(\mathbf{B}_1 + \mathbf{B}_1^T) + \frac{1}{2}(\mu - \frac{1}{\mu})(\mathbf{B}_1 - \mathbf{B}_1^T)\,. \tag{13}$$

For the physically interesting case when the twist $\mathbf{B}_1$ is symmetric, the last term vanishes, and the spectrum is determined by the generalized eigenvalue problem

$$\det(\mathbf{M}_1(1) - 4R\mathbf{B}_1) = 0\,. \tag{14}$$

Since $\mathbf{M}_1(1)$ is positive definite, and both matrices are symmetric, they can be simultaneously diagonalized. Thus the residue is obtained as the eigenvalue of a symmetric matrix and therefore must be real, ruling out complex quadruplets of multipliers. Elliptic multipliers are possible for arbitrary symmetric $\mathbf{B}$, and occur when $0 < R < 1$.

However, if in addition $\mathbf{B}$ is negative definite, then elliptic multipliers cannot occur, because we can write

$$\mathbf{M}_1(e^{i\phi}) = \mathbf{M}_1(1) + 2(1 - \cos\phi)(-\mathbf{B}_1)\,. \tag{15}$$

This is positive definite since it is the sum of a positive definite matrix and a positive semidefinite matrix. Therefore, $\det \mathbf{M}_1(\exp(i\phi)) \neq 0$, and there are no multipliers on the unit circle. Thus for negative definite twist a nondegenerate mimimizing fixed point is hyperbolic in d dimensions.

If the twist $\mathbf{B}_1$ is symmetric but indefinite it is certainly possible to have elliptic multipliers, e.g., by choosing $\mathbf{M}_1(1)$ and $\mathbf{B}_1$ diagonal.

If the twist $\mathbf{B}_1$ is not symmetric the eigenvalue problem for R cannot be derived in this simple way. Introducing the symmetric part of the twist $\tilde{\mathbf{S}} = (\mathbf{B}_1 + \mathbf{B}_1^T)/2$ and its antisymmetric part $\tilde{\mathbf{Y}} = (\mathbf{B}_1 - \mathbf{B}_1^T)/2$ we can rewrite $\det(\mathbf{M}_1(\mu)) = 0$ as

$$\det(\mathbf{M}_1(1) - 4R\tilde{\mathbf{S}} - 4\delta\tilde{\mathbf{Y}}) = 0, \tag{16}$$

where $\delta = (1/\mu - \mu)/4 = \sqrt{R(1-R)}$. By simultaneous diagonalization we can again simplify the problem in reducing $\mathbf{M}_1(1)$ to the identity and $\tilde{\mathbf{S}}$ to the diagonal $\mathbf{S}$. $\mathbf{Y}$

denotes the transformed $\tilde{\mathbf{Y}}$ which is still antisymmetric, such that

$$\det(\mathbf{1} - 4R\mathbf{S} - 4\delta\mathbf{Y}) = 0. \tag{17}$$

We know that this must be a polynomial in R, because the reflexivity of the characteristic polynomial for the multiplier μ [3] allows it to be rewritten as a polynomial of degree d in $\mu + 1/\mu$, or, equivalently, in R. To see this explicitly we employ the "cumulant expansion" for an arbitrary $n \times n$ matrix $\mathbf{A}$

$$\det(\mathbf{1} + \varepsilon\mathbf{A}) = \sum_{i=0}^{n} \varepsilon^i Q_i(\mathbf{A}), \tag{18}$$

where the cumulants (or up to a sign the coefficients of the characteristic polynomial of $\mathbf{A}$) are recursively defined by

$$\begin{aligned} Q_0 &= 1, \\ Q_1 &= \operatorname{tr}\mathbf{A}, \\ Q_i &= \frac{1}{i}\sum_{k=1}^{i}(-1)^{k+1}Q_{i-k}(\mathbf{A})\operatorname{tr}\mathbf{A}^k. \end{aligned} \tag{19}$$

We apply this formula to $\det(\mathbf{1} + \epsilon(4R\mathbf{S} + 4\delta\mathbf{Y}))$ and eventually set $\epsilon = -1$. For large dimensions it is quite cumbersome to obtain explicit expressions for the "characteristic polynomial" of R because in the expansion of $\operatorname{tr}\mathbf{A}^k$ we must compute terms of the form

$$\operatorname{tr}(R\mathbf{S} + \delta\mathbf{Y})^k = \sum_{j+l=k} \rho^j \delta^l \operatorname{tr}(\sigma(\mathbf{S}, j, \mathbf{Y}, l)), \tag{20}$$

where $\sigma(\mathbf{S}, j, \mathbf{Y}, l)$ stands for the sum of all (noncommutative!) products with j factors $\mathbf{S}$ and l factors $\mathbf{Y}$ in all possible orderings. Since we can cyclically permute under the trace a lot of terms can be combined. Since in general the symmetric and the antisymmetric part of the twist do not commute, these expressions contain traces of products of $\mathbf{S}$ and $\mathbf{Y}$ for $k > 2$. For $d = 2, 3$ we obtain

$$0 = \det(\mathbf{1} - 4R\mathbf{S}) - 8R(1-R)\operatorname{tr}(\mathbf{Y}^2) \tag{21}$$

$$0 = \det(\mathbf{1} - 4R\mathbf{S}) - 8R(1-R)\left(\operatorname{tr}(\mathbf{Y}^2)(1 - 4R\operatorname{tr}\mathbf{S}) + 8R\operatorname{tr}(\mathbf{S}\mathbf{Y}^2)\right) = 0. \tag{22}$$

Now we argue that all the terms with an odd number of $\mathbf{Y}$ vanish. Consider an arbitrary term with an odd number of $\mathbf{Y}$ in the sequence of $\mathbf{S}$ and $\mathbf{Y}$. If reading the sequence backwards is the same sequence, then this term is antisymmetric and its trace vanishes. If reading the sequence backwards gives another sequence, then this sequence is also part of the sum, and their sum is antisymmetric, hence vanishes under the trace. Therefore Equations (16) and (17) define polynomials in R of degree d.

If $\mu = 1$ then $\delta = 0$ and $R = 0$ such that the general determinant (17) can never vanish. This means that a minimizing orbit can not undergo a saddle node bifurcation (without loosing the minimizing property). If $\mu = -1$ then again $\delta = 0$ but now $R = 1$. Therefore $\det(\mathbf{1}+4\mathbf{S}) = 0$, and since $\mathbf{S}$ is diagonal one of its eigenvalues must be $1/4$. Note that this condition for a period doubling bifurcation is independent of the antisymmetric part of the twist. In [5] a similar condition for a period doubling bifurcation of (not only minimizing) fixed points of natural maps is obtained.

For $d = 2$ we can perform a more detailed analysis by determining the residues in the space of three essential parameters $\mathbf{S} = \mathrm{diag}(d_1, d_2)$ and a, the single entry of the antisymmetric $\mathbf{Y}$. The polynomial determining ρ is given by (17) resp. (21), or more explicitly

$$
\begin{aligned}
0 &= 16R^2 \det \tilde{\mathbf{S}} + 4R \,\mathrm{tr}(\tilde{\mathbf{S}} \,\mathrm{adj}\, \mathbf{M}) + \det \mathbf{M} - 4R(1-R)\,\mathrm{tr}(\tilde{\mathbf{Y}}^2) &\quad (23)\\
&= (4d_1R - 1)(4d_2R - 1) + 16a^2R(R-1). &\quad (24)
\end{aligned}
$$

adj denotes the matrix of cofactors, i.e. the inverse of the matrix times its determinant if it is nonsingular. Its roots pass through infinity if $d_1d_2 + a^2 = 0$; they are complex if $16a^4 + 4a^2(2(d_1 + d_2) - 1) + (d_1 - d_2)^2 < 0$. As in the general case $R = 0$ is impossible and $R = 1$ corresponds to $d_i = 1/4$. For $a = 0$ the plane (d_1, d_2) is therefore divided into 9 regions by the 4 lines $d_i = 0, 1/4$, see Figure 1. For negative definite symmetric twist we have $d_i < 0$ and the negative quadrant corresponds to multipliers of type HH. The transition from HH to any region in the adjacent quadrants is not a regular bifurcation, because it induces R to pass through infinity. In a smooth system this is impossible. Basically it means that the signature of the symmetric twist is preserved under smooth parameter variation, which is by definition true in the general case. If the signature of the twist is mixed, we have HE or HI, and if it is positive, then we have II, IE or EE, the transitions taking place at $d_i = 1/4$. Now making a nonzero can not change EE or IE,

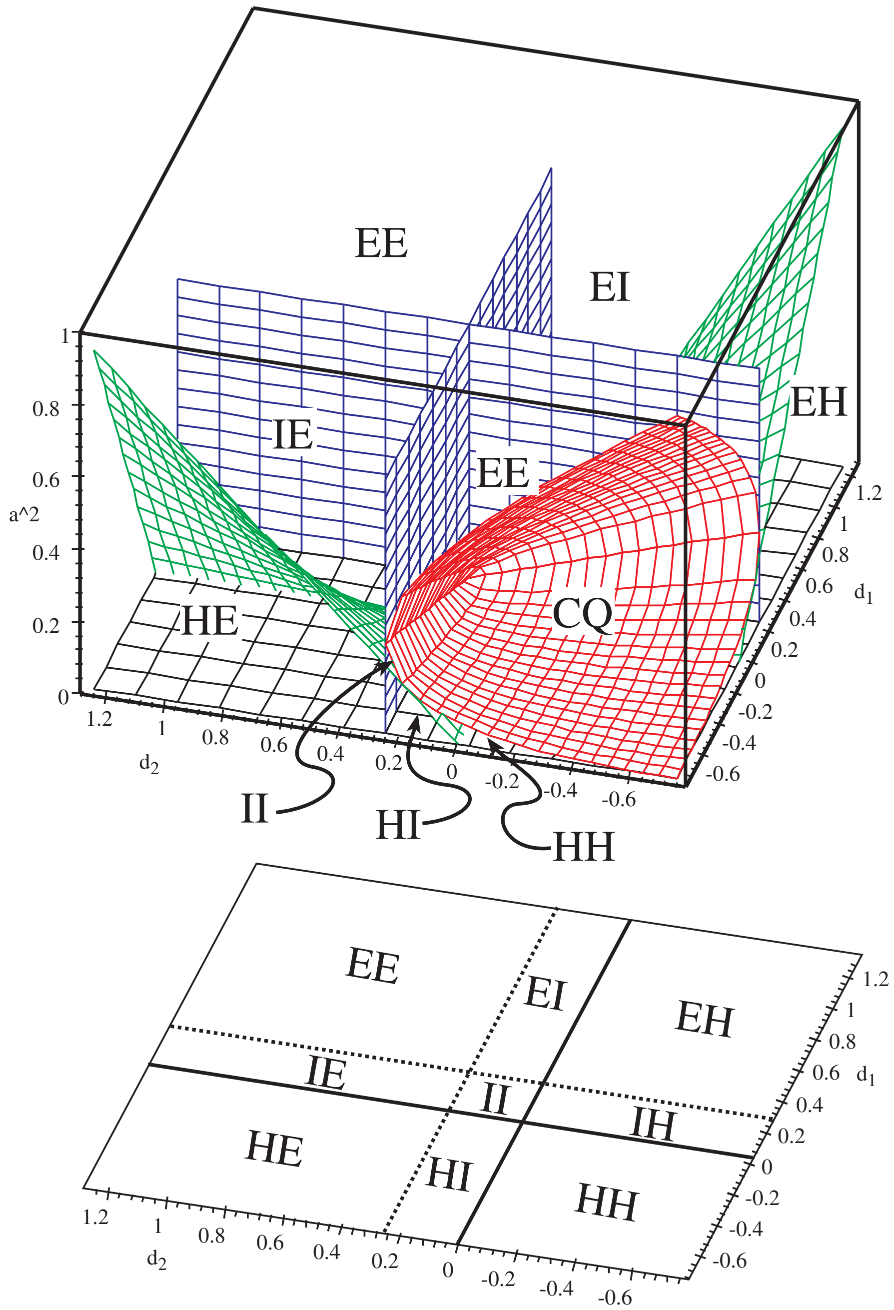

Figure 1: Stability of a minimizing orbit for a 4D map in the space of the three essential parameters d_1, d_2 and the square of the antisymmetry of the twist a^2.

see Figure 1. Similarly we can not change HE or HI, because this would involve driving R through infinity, i.e. making the twist singular.

The main change occurs above the HH and II region. Increasing a leads to a complex bifurcation where four real multipliers collide and turn complex, entering the region CQ. Increasing a further leads to the inverse complex bifurcation in which four complex multipliers collide on the unit circle, hence creating four elliptic multipliers. Since these elliptic multipliers are created in a complex bifurcation their Krein signatures must be different. Note that even though it looks like the two EH regions are disconnected, this is due to the ambiguity in the ordering of the eigenvalues in $\mathbf{S}$. In the full parameter space they are connected and together with IH form a region bounded by $\det \mathbf{M}_1(1) = 0$. All the other regions are smoothly connected; only for symmetric twist the HH region is separated from the others.

The main result for $d = 2$ therefore is that for negative definite twist a sufficiently large antisymmetric part can turn the minimizing hyperbolic fixed point elliptic via an (inverse) Krein collision.

3 Periodic Orbits

We now turn to the calculation of stability of periodic orbits. For period two we explicitly construct $\mathbf{M}_1^{(2)}(\mu)$ for the iterated map F^2 and then make the connection with Schur's complement [12] of $\mathbf{M}_2$, in order to show that the product can be generated by a twist generating function of a minimal fixed point. For $n > 2$ we will directly work with Schur's complement to establish this result. Recall that the Schur complement $(\mathbf{M}|\mathbf{D})$ of $\mathbf{M}$ with respect to $\mathbf{D}$ is defined by the following factorization

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} (\mathbf{M}|\mathbf{D}) & \mathbf{B}\mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \tag{25}$$

such that

$$(\mathbf{M}|\mathbf{D}) = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}. \tag{26}$$

$\mathbf{A}$ and $\mathbf{D}$ are square matrices; if they have different dimensions then $\mathbf{B}$ and $\mathbf{C}$ are not square matrices. The factorization of $\mathbf{M}$ gives a factorization of its determinant,

$$\det \mathbf{M} = \det(\mathbf{M}|\mathbf{D}) \det \mathbf{D}. \tag{27}$$

We will need the fact [12] that the Schur complement of a symmetric positive definite matrix is symmetric and positive definite. This is easily seen because transforming the quadratic form corresponding to $\mathbf{M}$ with

$$\mathbf{T} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{D}^{-1}\mathbf{B}^t & \mathbf{1} \end{pmatrix} \quad \text{gives} \quad \mathbf{T}^t\mathbf{M}\mathbf{T} = \mathrm{diag}((\mathbf{M}|\mathbf{D}), \mathbf{D}). \tag{28}$$

For a periodic orbit of period $n = 2$ we could multiply $DF(x_2, y_2)$ and $DF(x_1, y_1)$, and identify the resulting matrix to be of the form (5). It is simpler to consider the 2nd difference equation for the period 2 orbit,

$$\begin{aligned} \mathbf{B}_1^T\delta x_1 + \mathbf{P}_2\delta x_2 + \mu\mathbf{B}_2\delta x_1 &= 0 && (29) \\ \frac{1}{\mu}\mathbf{B}_2^T\delta x_2 + \mathbf{P}_1\delta x_1 + \mathbf{B}_1\delta x_2 &= 0. && (30) \end{aligned}$$

Solving the first equation for δx_2 and eliminating it in the second directly gives $\mathbf{M}_1^{(2)}$. The superscript 2 denotes that the matrix is that of a fixed point corresponding to a period 2 orbit. By comparsion with (8) we find

$$\begin{aligned} \mathbf{B}_1^{(2)} &= -\mathbf{B}_1\mathbf{P}_2^{-1}\mathbf{B}_2 && (31) \\ \mathbf{P}_1^{(2)} &= \mathbf{P}_1 - \mathbf{B}_2^T\mathbf{P}_2^{-1}\mathbf{B}_2 - \mathbf{B}_1\mathbf{P}_2^{-1}\mathbf{B}_1^T. && (32) \end{aligned}$$

$\mathbf{B}_1^{(2)}$ and $\mathbf{P}_1^{(2)} = \mathbf{A}_1^{(2)} + \mathbf{D}_1^{(2)}$ define a generating function by (4) for the iterated map. The splitting of $\mathbf{P}_1^{(2)}$ into $\mathbf{A}_1^{(2)}$ and $\mathbf{D}_1^{(2)}$ is arbitrary for our purposes; only $\mathbf{P}_1^{(2)}$ enters the stability formulae.

Our task is to show that the fact that the periodic orbit is minimal, $\mathbf{M}_2(1) > 0$, implies that the new twist $\mathbf{B}_1^{(2)}$ is nonsingular and that $\mathbf{M}_1^{(2)}(1) > 0$. $\det \mathbf{B}_1^{(2)} \neq 0$ because $\det \mathbf{B}_i \neq 0$ by assumption and $\det \mathbf{P}_2 \neq 0$ is implied by $\mathbf{M}_2(1) > 0$, because $\mathbf{P}_2$ is a principal subblock[1] of $\mathbf{M}_2$.

[1]By principal subblock we mean a block that is centered on the diagonal.

To show that $\mathbf{M}_1^{(2)}(1) > 0$ we note that

$$\begin{aligned}
\mathbf{M}_1^{(2)}(1) &= \mathbf{P}_1^{(2)} + \mathbf{B}_1^{(2)} + (\mathbf{B}_1^{(2)})^T && (33)\\
&= \mathbf{P}_1 - (\mathbf{B}_1 + \mathbf{B}_2^T)\mathbf{P}_2^{-1}(\mathbf{B}_1^T + \mathbf{B}_2) && (34)\\
&= (\mathbf{M}_2(1)\,|\,\mathbf{P}_2)\,. && (35)
\end{aligned}$$

Now the desired statement immediately follows because the Schur complement of a symmetric positive definite matrix is again symmetric positive definite.

Instead of repeating this calculation for the general case of period n we directly use Schur's complement on the matrix $\mathbf{M}_n(\mu)$ to recursively reduce dimension by d in each step. The final result after $n-1$ steps is $\mathbf{M}_1^{(n)}(\mu)$ where the superscript is an iteration index. From this we can identify the twist $\mathbf{B}_1^{(n)}$ of the generating function for the nth iterate of the map in the neighborhood of the minimal period n orbit via (8). The proof proceeds by induction. The initial matrix gets the iteration index 1, $\mathbf{M}_n(\mu) \equiv \mathbf{M}_n^{(1)}(\mu)$. The iteration rule is

$$\mathbf{M}_{k-1}^{(i+1)}(\mu) = \left(\mathbf{M}_k^{(i)}(\mu)\,|\,\mathbf{P}_k^{(i)}\right), \tag{36}$$

or more explicitly

$$\begin{aligned}
\mathbf{P}_1^{(i+1)} &= \mathbf{P}_1^{(i)} - (\mathbf{B}_k^{(i)})^T(\mathbf{P}_k^{(i)})^{-1}\mathbf{B}_k^{(i)} && (37)\\
\mathbf{P}_{k-1}^{(i+1)} &= \mathbf{P}_{k-1}^{(i)} - \mathbf{B}_{k-1}^{(i)}(\mathbf{P}_k^{(i)})^{-1}(\mathbf{B}_{k-1}^{(i)})^T && (38)\\
\mathbf{B}_{k-1}^{(i+1)} &= -\mathbf{B}_{k-1}^{(i)}(\mathbf{P}_k^{(i)})^{-1}\mathbf{B}_k^{(i)} && (39)\\
\mathbf{B}_j^{(i+1)} &= \mathbf{B}_j^{(i)}, \quad j = 1, \ldots, k-2 && (40)\\
\mathbf{P}_j^{(i+1)} &= \mathbf{P}_j^{(i)}, \quad j = 2, \ldots, k-2 && (41)
\end{aligned}$$

The last two lines merely state that these entries do not change, while the matrices $\mathbf{P}_k^{(i)}$ and $\mathbf{B}_k^{(i)}$ are discarded in reducing the dimension by d. Note that for $k = 2$ these formulas collapse to (31). Parts of this iteration formula are identical to those reported in [9] and [6]. The formulation we have choosen here allows us to reduce minimal periodic orbits of twist maps to fixed points of twist maps. This fact has not been realized before, and we are now going to prove it.

Since we start with a positive definite matrix $\mathbf{M}_n^{(1)}(1) > 0$, the next iterate constructed by Schur's complement is also positive definite. By induction all $\mathbf{M}_{n-i}^{(i+1)}(1) > 0$. By assumption $\mathbf{B}_n^{(1)}$ and $\mathbf{B}_{n-1}^{(1)}$ in $\mathbf{M}_n^{(1)}$ are nonsingular, and since $\mathbf{P}_n^{(1)}$ is a principal subblock of the positive definite matrix $\mathbf{M}_n^{(1)}$ it is positive definite, and therefore also nonsingular. In the iteration step from i to $i+1$, $k = n - i + 1$, one of the relevant twist matrices is not changed, $\mathbf{B}_{k-2}^{(i+1)} = \mathbf{B}_{k-2}^{(i)}$, the other one obtained from (39) is also nonsingular because by assumption 1) the two matrices on the right hand side of (39) are nonsingular, and 2) the matrix $\mathbf{P}_k^{(i)}$ in the same equation is nonsingular because it is a principal subblock of a positive definite matrix. In conclusion we have shown that the twist stays nonsingular and that the matrices $\mathbf{M}_{n-i}^{(i+1)}(1)$ stay positive definite.

Although in general the composition of twist maps does not give a twist map we have shown that in the neighborhood of a minimal period n orbit there exists a local generating function with nonsingular twist for the n times iterated map. The essential observation concerning stability of minimal periodic orbits is that the property of having symmetric negative definite twist is *not* stable under this iteration. The final twist is given by

$$\mathbf{B}_1^{(n)} = \mathbf{B}_1 \prod_{i=2}^{n} (\mathbf{P}_i^{(n-i+1)})^{-1}(-\mathbf{B}_i). \tag{42}$$

So even in the case of higher dimensional standard maps $L(x, x') = (x' - x)^2/2 - U(x)$ which have constant symmetric negative definite twist $\mathbf{B}_i = -\mathbf{1}$, for $n > 2$ we obtain the product of $n-1$ symmetric positive definite matrices which is in general neither symmetric nor positive definite. However, if the matrices $\mathbf{P}_i^{(n-i+1)}$ commute with each other then their product is symmetric and positive definite. This can be achieved if the Hessian of the potential $U(x)$ is diagonal for all x; then these matrices are diagonal and therefore commute. But this is true only if the potential separates, such that we are back to the case $d = 1$.

Note that if we apply the determinant formula for Schur's complement (27) to the iteration rule (36) we obtain

$$\det \mathbf{M}_{k-1}^{(i+1)}(\mu) = \det \left(\mathbf{M}_k^{(i)}(\mu) \,|\, \mathbf{P}_k^{(i)} \right) \det \mathbf{P}_k^{(i)}. \tag{43}$$

In each step the last factor is nonzero, such that we can ignore all of them and find

$$0 = \det \mathbf{M}_n^{(1)}(\mu) \quad \Longleftrightarrow \quad 0 = \det \mathbf{M}_1^{(n)}(\mu), \tag{44}$$

where on the left we have the determinant of an $nd \times nd$ matrix, while on the right it is only a $d \times d$ matrix. This therefore gives an efficient way to calculate the multipliers of a minimal periodic orbit from the Hessian of the periodic action.

Finally we give two examples of minimizing periodic orbits that are not hyperbolic. The first example is a little artificial since it involves a nonconstant twist. The second, however, shows that in 4D natural maps [5] minimizing period three orbits can be elliptic. The first example in $d = 2$ dimensions and for period $n = 2$ is given by

$$\mathbf{P}_1 = \mathrm{diag}(5/3, 3/4), \quad \mathbf{P}_2 = \mathrm{diag}(2/3, 3/2) \tag{45}$$

and

$$\mathbf{B}_1 = \begin{pmatrix} -1/2 & b \\ b & -1/2 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} -1/2 & -b \\ -b & -1/2 \end{pmatrix}, \tag{46}$$

which are symmetric negative definite twist matrices as long as $|b| < 1/2$. The resulting matrix $\mathbf{M}_2(1)$ is positive definite. However, the resulting multipliers can be either HH, CQ, EE, or EH depending on the value of b. In particular for $b = 1/3$ the eigenvalues of $\mathbf{M}_2(1)$ are given by

$$(9\lambda^2 - 21\lambda + 1)(8\lambda^2 - 18\lambda + 1) = 0, \tag{47}$$

which are all positive, while the multipliers are given by

$$(5\mu^2 + 2\mu + 5)(10\mu^2 - 17\mu + 10) = 0, \tag{48}$$

which all have modulus one. Let's now turn to the case of 4D natural maps. Note that by the above we have shown that their minimal period 2 orbits are hyperbolic, because in this case (42) contains only one term. To construct an example with elliptic eigenvalues for $\mathbf{B}_i = \mathrm{diag}(-1, -1)$ we have to go to period 3, e.g. by choosing

$$\mathbf{P}_1 = \begin{pmatrix} 6.2 & -1.1 \\ -1.1 & 1.7 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} 0.4 & 0 \\ 0 & 1.8 \end{pmatrix}, \mathbf{P}_3 = \begin{pmatrix} 7.2 & 2 \\ 2 & 3.8 \end{pmatrix} \tag{49}$$

Since the eigenvalues of some $\mathbf{P}_i$ have to be quite different in magnitude in order to produce this effect, we have not been able to find a minimizing period three orbit of the Froeschlé map that shows this phenomenon. But obviously one can construct a potential of a natural 4D map which supports the above period 3 orbit.

4 Discussion

We have shown that periodic orbits with minimal action of a twist map with negative definite twist can be elliptic. This result was obtained in three stages. First we showed that a non-degenerate fixed point with minimal action of a twist map with negative definite symmetric twist is hyperbolic in any number of dimensions. For application to the case of periodic orbits we noted that if the twist is not definite or not symmetric then this is not true. Arnaud [1] showed that there exist twist maps that have no hyperbolic fixed points. This implies that there exist maps with minimizing fixed points that are not hyperbolic, provided the Lagrangian is bounded from below. Our main point was to show that there exist maps whose minimal fixed points are hyperbolic, but nevertheless the minimal periodic orbits are not.

In order to show this, we first derived the interesting result that in the neighborhood of a minimal periodic orbit the iteration of a twist map is again a twist map, which is not always true globaly. The key to this observation was the use of Schurs complement to recursively reduce the dimension of the Hessian of the periodic action. Starting with an $nd \times nd$ dimensional Hessian of a period n orbit of a d dimensional map the final result is a $d \times d$ matrix, which can be interpreted as the Hessian of the fixed point of a Lagrangian map.

This reduction in the neighborhood of a minimal periodic orbit allowed us to show that these periodic orbits can be non-hyperbolic. The main point is that in the reduction process the property to have symmetric negative definite twist can be destroyed, and therefore the orbit can be non-hyperbolic. Arnaud's results [2] gives a similar statement for flows: there exist autonomous Lagrangian flows for which the minimizing orbits are not hyperbolic. Our result for maps complements her result for autonomous flows because it amounts to treating Lagrangians with explicit time dependence.

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