Heteroclinic primary intersections and codimension one Melnikov method for volume preserving maps *

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September 10, 1999

Abstract

We study families of volume preserving diffeomorphisms in $\mathbb{R}^3$ that have a pair of hyperbolic fixed points with intersecting codimension one stable and unstable manifolds. Our goal is to elucidate the topology of the intersections and how it changes with the parameters of the system. We show that the "primary intersection" of the stable and unstable manifolds is generically a neat submanifold of a "fundamental domain". We compute the intersections perturbatively using a codimension one Melnikov function. Numerical experiments show various bifurcations in the homotopy class of the primary intersections.

PACS, 05.45.Ac, 45.20Jj, 47.52.+j

The theory of transport for area preserving maps is based on the construction of "partial barriers," typically from segments of stable and unstable manifolds of fixed points, periodic or quasiperiodic orbits. Our ultimate goal is the generalization of this theory to higher dimensions. Perhaps the simplest place to start is with volume preserving maps in three dimensions. A hyperbolic fixed point of such a map has either a two dimensional stable or unstable manifold. Since they are codimension one, these manifolds can separate phase space into regions containing nontrivial invariant sets. The major problem is to choose appropriate domains of these manifolds that can be used in the construction of partial barriers. To this end we define "fundamental domains" and their "primary intersections" by using a partial ordering along the manifolds. Primary intersections are typically curves on the two dimensional manifolds. These curves,

*Useful conversations with R. Easton are gratefully acknowledged. JDM was supported in part by NSF grant number DMS-9971760.

when restricted to a fundamental domain, become loops and can be classified by their homotopy. As parameters of a map change, these homotopy classes can change as well. To investigate this, we start with an integrable map that has a saddle connection, and use a Melnikov perturbation method to compute the splitting distance between the manifolds. Our numerical computations show the creation and destruction of intersection loops of various types.

1 Introduction

Volume preserving maps provide an interesting and nontrivial class of dynamical systems that give perhaps the simplest, natural generalization of the class of area preserving maps to higher dimensions. Moreover, volume preserving maps naturally arise in applications as the time one Poincaré map of incompressible flows—even when the vector field of the flow is nonautonomous. Thus the study of the dynamics of volume preserving maps is of interest both for fluids and magnetic fields. Our primary motivation for studying volume preserving maps is to generalize the study of transport, which has been quite successful for two dimensional maps [1, 2], to higher dimensional cases.

Previously, we constructed a normal form for the quadratic volume preserving map in $\mathbb{R}^3$ [3] (and obtained a partial classification for higher dimensions [4]). This map is the natural generalization of Hénon's quadratic area preserving map [5]—it gives the simplest volume preserving system in $\mathbb{R}^3$ that has nontrivial dynamics. Moser has similarly obtained normal forms for quadratic symplectic maps [6]. These normal forms are to be distinguished from formal series expansions that give normal forms in the neighborhood of a fixed point, such as the "Birkhoff" normal form. Bazzani has constructed such normal forms for volume preserving maps, showing that they are formally integrable to all orders [7]. These normal forms are formal series expansions that typically do not converge, and moreover are not volume preserving when truncated at any given finite order.

The quadratic volume preserving map has at most two fixed points, and typically these points are hyperbolic and have either a two dimensional stable and a one dimensional unstable manifold (type A), or a two dimensional unstable and a one dimensional stable manifold (type B). Commonly one point is type A and the other type B, and the two dimensional stable manifold of the first intersects the two dimensional unstable manifold of the second. In this paper we investigate the properties of such intersections.

An understanding of the intersections of codimension one invariant manifolds is important in the development of transport theory [8, 9, 10]. For example, suppose a and b are saddle fixed points of an area preserving map, and $W^s(a)$ and $W^u(b)$ are branches of their stable and unstable manifolds. If these intersect at a point x, then x is a heteroclinic orbit; that is, it is backward asymptotic to a and forward asymptotic to b. Let $W_x(a)$ denote the segment of an invariant manifold that starts at a and extends to x (below we will be careful to exclude or include endpoints of these segments as appropriate). A point x is a *primary* intersection point (or "pip" in Wiggins' terminology [2]) if the closures of $W^s_x(a)$ and $W^u_x(b)$ intersect only at the endpoint x.

Primary intersections can be used to form *resonance zones* [11, 12]—regions of phase space that are bounded by alternating stable and unstable segments joined at primary intersection points. Because the intersection points are primary, a resonance zone is bounded by a Jordan curve, and it has an exit and an entry set [13]. The images of these sets completely define the transport properties of the resonance zone.

In §3 we generalize the notion of primary intersections to $\mathbb{R}^3$. A similar theory was developed in [14] for the case of diffeomorphisms that arise from quasiperiodic time-dependent vector fields, see [2] for a review. Our definition depends on the existence of two dimensional manifolds and our main concern is to understand the topology of the one dimensional intersection as an immersed submanifold. It is possible that the stable and unstable manifolds could be used to construct partial barriers, and their intersections will bound "lobes" that can be used to compute transport properties [15]. We will see that different homotopy classes of primary intersections can exist and that they can bifurcate by changing from one homotopy class to another. Bifurcations in the intersection manifolds will have immediate consequences for transport, since such a bifurcation modifies the lobes, and may even forbid their existence. Our definition can be generalized to higher dimensions, if the map has codimension one invariant manifolds.

In order to illustrate how the heteroclinic intersection changes, we develop an extension of the Melnikov method to volume preserving maps in §4. Melnikov methods have been extensively developed for two dimensional maps [16, 17, 18], for higher dimensional maps [19, 20] and for three dimensional volume preserving flows [21]. In this latter case the perturbation may be periodically time dependent, and the Poincaré map of the system is assumed to have a hyperbolic invariant curve, with two dimensional manifolds.

For the case of maps, the analogue of Melnikov integral is an infinite sum whose domain is the unperturbed connection. As usual, a simple zero of this function corresponds to a transverse intersection for the perturbed map.

In §5 we introduce a family of volume preserving maps that have a completely degenerate heteroclinic connection (i.e, a saddle connection). This family is obtained from a family of planar twist maps with saddle connections [22]. We perturb the family by composing it with a near identity, volume preserving map. In this way, we can produce examples of volume preserving maps with transverse heteroclinic orbits. To accomplish this construction we will need a pair of *adapted vector fields* on the manifold, or alternatively, an integral of the unperturbed map.

We compute the Melnikov function in §6 for a particular perturbation and a classify the primary intersection curves by their homotopy type. We observe a number of bifurcations as the parameters of the map change. To compare the Melnikov function with the fully nonlinear map, we compute its stable and unstable manifolds. In general, the development of computational methods for the effective visualization of invariant manifolds in higher dimensional maps is itself an interesting and difficult problem [23, 24]. For the case that the magnitude of the multipliers at the fixed point restricted to the unstable subspace are equal, we apply a clever, but simple technique due to Tabacman [25].

2 Volume preserving maps

A diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is volume preserving when $f^*\Omega = \Omega$, where Ω is a volume form, i.e., a positive n-form. We will restrict our consideration to maps on $\mathbb{R}^3$ that preserve the standard volume form

$$\Omega = dx \wedge dy \wedge dz. \tag{1}$$

In this case the volume preserving condition reduces to $\det(f') = 1$, i.e., that the Jacobian of f is one. For later reference, we recall that if $v_1, v_2, v_3 \in T_\xi(\mathbb{R}^3)$ then

$$\Omega(v_1, v_2, v_3) = \det(v_1, v_2, v_3) = < v_1, v_2 \times v_3 > . \tag{2}$$

where $<,>$ is the standard inner product. We also know that the triple product satisfies

$$\Omega(v_1, v_2, v_1 \times v_2) = |v_1 \times v_2|^2. \tag{3}$$

Volume preserving maps arise naturally in connection with divergence free vector fields. If X is a vector field and Ω is a volume form, then the divergence of X with respect to Ω is defined as the unique differentiable function $\operatorname{div} X$ such that $L_X\Omega = (\operatorname{div} X)\Omega$. Thus the time t map of any divergence free flow is volume preserving, and such maps arise naturally from the time one maps of incompressible, time-dependent fluid or magnetic field line flows.

For a volume preserving map, the multipliers of any periodic orbit must have a product one. For example, suppose p is a fixed point of the volume preserving map f on $\mathbb{R}^3$. Then $\det(f'(p)) = 1$ and therefore the characteristic polynomial of the Jacobian matrix $f'(p)$ is

$$\lambda^3 - t\lambda^2 + s\lambda - 1 = 0 \ ,$$

where t is the trace and s is the so called *second trace* of $f'(p)$.

The dependence of the multipliers on the two parameters t and s is illustrated in Fig. 1. There are two lines in parameter space where the stability changes: the saddle-node line, $t = s$, corresponds to an eigenvalue 1, and the period doubling line, $t+s = -2$, corresponds to an eigenvalue -1. At the point $t = s = -1$ where they cross the multipliers are necessarily $(-1, -1, 1)$. Note also that when $-1 \leq t = s \leq 3$ there is a pair of multipliers on the unit circle. There are two other curves of interest–these correspond to a double eigenvalue $\lambda_1 = \lambda_2 = r$, or

$$t = 2r + 1/r^2 \quad s = r^2 + 2/r \ .$$

This gives the two curves shown in Fig. 1. One has a cusp at $t = s = 3$, corresponding to the triple root $\lambda = 1$. The second crosses the saddle-node and period doubling lines at $t = s = -1$. These are codimension two points.

A hyperbolic fixed point with a two dimensional stable manifold is called *type* A and one with a two dimensional unstable manifold is called *type* B [26]. These fixed points also have one dimensional unstable and stable manifolds, respectively. The saddle-node and period doubling lines divide the (t, s) plane into quadrants which alternate between type A and B. The dynamics on the two dimensional manifolds will depend upon whether the pair of multipliers are complex or are real.

If a map has a pair of fixed points one of type A and one of type B and the pair of two dimensional manifolds (stable and unstable) intersect, then generically they intersect along one dimensional manifolds. We have observed earlier [3] changes in the topology of the intersection manifolds as the parameters vary. Elucidating this topology is the primary aim of this paper.

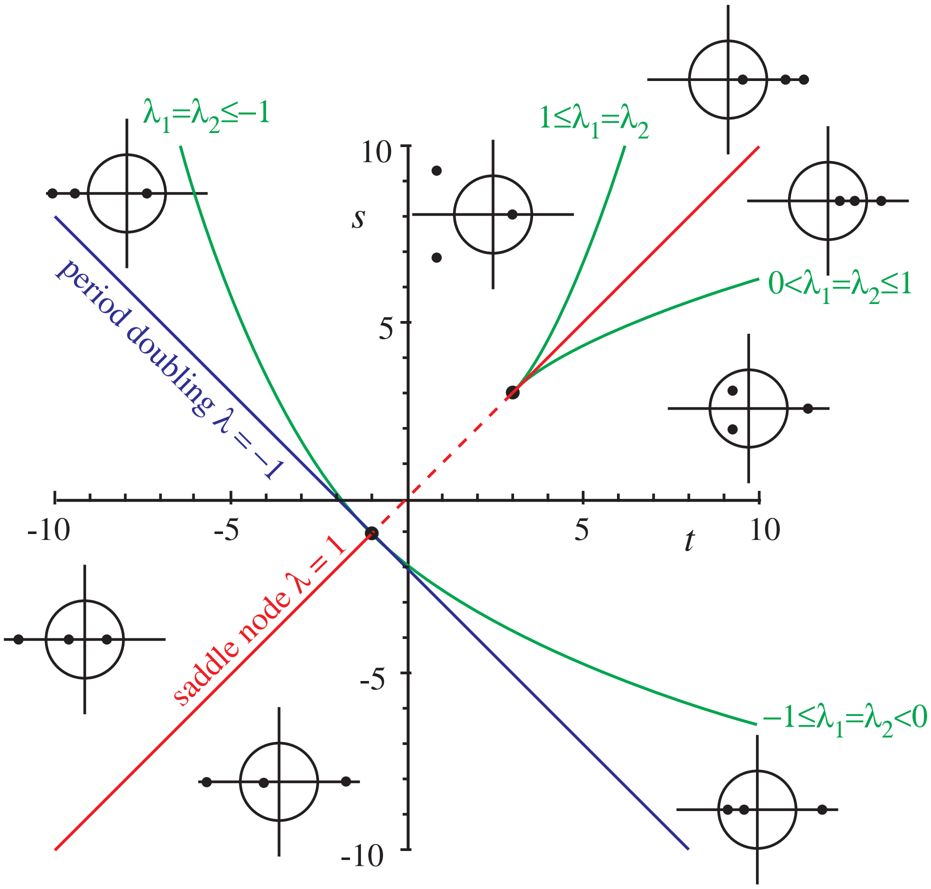

Figure 1: General stability diagram for a volume preserving map.

3 Primary Intersections

In this section we introduce the concepts of the fundamental domain of a stable (or unstable) manifold and of primary heteroclinic intersections between such manifolds. These generalize the well known concepts for two dimensional maps. We, as usual, assume that $f : \mathbb{R}^3 \to \mathbb{R}^3$ preserves the 3-form Ω, (1).

Proper Loops and Fundamental Domains

Definition 1 (Proper Loop) *Suppose $a = f(a)$ is hyperbolic and of type* A, *i.e., has a two dimensional stable manifold $W^s(a)$. A proper loop $\gamma \subset W^s(a)$ is a curve that bounds a local submanifold that is an isolating neighborhood of a. In other words γ is proper if there is an open local submanifold $W^s_{loc}(a)$ such that*

a) $\partial W^s_{loc}(a) = \gamma$ *and*

b) $f(\overline{W^s_{loc}(a)}) \subset int(W^s_{loc}(a))$.

Similarly if b is a type B *fixed point, then a loop $\sigma \subset W^u(b)$ is proper it is proper for f^{-1}.*

If γ is proper, we can define the stable manifold *starting at* γ, denoted by by $W^s_\gamma(a)$, as the closure in $W^s(a)$ of the local submanifold bounded by γ in Defn. 1. Similarly, if b is a type B fixed point with a proper loop σ, we define the manifold *up to* σ, denoted $W^u_\sigma(b)$, as the interior of the local manifold that corresponds to f^{-1} in Defn. 1.

Notice that the definition is not symmetric, because $W^s_\gamma(a)$ is a closed subset of $W^s(a)$, while $W^u_\sigma(b)$ is open in $W^u(b)$ (cf. Fig. 2). The asymmetry is just a technicality in order to simplify some proofs.

Definition 2 (Fundamental domain) *Let a and b be hyperbolic fixed points of type* A *and* B, *respectively. An annulus $\mathcal{S}$ is a* fundamental domain *of $W^s(a)$ if there exists some proper loop γ in $W^s(a)$, such that*

$$\mathcal{S} = W^s_\gamma(a) \setminus W^s_{f\circ\gamma}(a) \ .$$

Similarly, a fundamental domain in $W^u(b)$ is a manifold with boundary of the form

$$\mathcal{U} = W^u_\sigma(b) \setminus W^u_{f^{-1}\circ\sigma}(b) \ ,$$

where σ is a proper loop in $W^u(b)$. In addition, we define $\mathcal{F}^u(b)$ as the set of all fundamental domains in $W^u(b)$, and $\mathcal{F}^s(a)$ as the set of all fundamental domains in $W^s(a)$.

In each case, the fundamental domain is an annulus with one open and one closed edge. An immediate consequence of the definition is that all the forward and backward iterations of a fundamental domain are also fundamental. It is easy to see that proper loops always exist, and in fact, the stable (and unstable) manifolds can be decomposed as the disjoint union of fundamental domains:

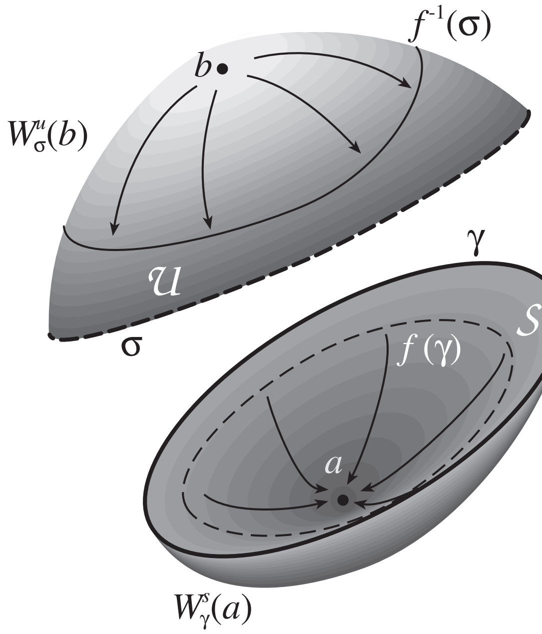

Figure 2: Proper loops γ and σ and corresponding fundamental domains $\mathcal{S}$ and $\mathcal{U}$ for fixed points a of type A and b of type B.

$$W^s(a) = \bigcup_{k\in\mathbb{Z}} f^k\left(\mathcal{S}_\gamma(a)\right).$$

The importance of fundamental domains is that much of the information about the entire manifold can be found by looking only at these annuli. For instance, the *primary heteroclinic intersection* between $W^s(a)$ and $W^u(b)$, which we define next, is defined using fundamental domains.

Primary Intersection

Given a fundamental domain $\mathcal{S}$, the points $\xi \in W^s(a)$ are given a partial order defined by the integer k such that $\xi \in f^k(\mathcal{S})$. This partial order provides an index that can be used to study heteroclinic intersections between two such manifolds:

Lemma 1 *Suppose $W^s(a) \cap W^u(b) \neq \emptyset$. Then for all $\mathcal{S} \in \mathcal{F}^s(a)$ and $\mathcal{U} \in \mathcal{F}^u(b)$, there exists a unique integer κ, called the* intersection index *such that*

$$\kappa(\mathcal{U},\mathcal{S}) \equiv \sup\{k \in \mathbb{Z} : \mathcal{U} \cap f^k(\mathcal{S}) \neq \emptyset\} = \sup\{k \in \mathbb{Z} : f^{-k}(\mathcal{U}) \cap \mathcal{S} \neq \emptyset\}.$$

Proof: This follows from the facts that each manifold is composed of the union of the fundamental domains, that the closures of $\mathcal{U}$ and $\mathcal{S}$ are compact and do not contain the fixed points, and that $f^k(\mathcal{S}) \to a$ and $f^{-k}(\mathcal{U}) \to b$ as $k \to \infty$. The two definitions are equivalent, since $f^{-k}\left(\mathcal{U} \cap f^k(\mathcal{S})\right) = f^{-k}(\mathcal{U}) \cap \mathcal{S}$. ■

The intersection index is useful because it is invariant: $\kappa(f(\mathcal{U}), f(\mathcal{S})) = \kappa(\mathcal{U}, \mathcal{S})$. More generally, the intersection index of iterates of fundamental domains changes as

$$\kappa(f^m(\mathcal{U}), f^n(\mathcal{S})) = \kappa(\mathcal{U}, \mathcal{S}) + m - n \; .$$

Roughly speaking, a primary intersection is the set of points where the stable and unstable manifolds "first" meet. For maps of the plane, one says that $x \in W^s(a) \cap W^u(b)$ is a primary intersection point if the intersection of the stable manifold starting at x and the unstable manifold up to x is empty: $W^s_x(a) \cap W^u_x(b) = \emptyset$. This means that one can choose fundamental domains $\mathcal{S}$ and $\mathcal{U}$ so that that their boundaries are (primary) heteroclinic points. As noted by Easton, this leads to a classification of heteroclinic orbits by their "type" [27], and subsequently a classification of the structure of the "trellis", the closure of the stable and unstable manifolds.

To directly generalize the planar definition, we would need to find a proper loop $\gamma = \sigma$ that is also heteroclinic, and such that $W^s_\gamma(a) \cap W^u_\gamma(b) = \emptyset$. These proper loops would be the analog of primary intersections. However, such loops need not exist as we saw in [3]. One consequence is that if one fixes a pair of fundamental domains $\mathcal{U}$ and $\mathcal{S}$, then the set of points at which $f^k(U)$ first intersects $\mathcal{S}$ is not necessarily a union of submanifolds of $\mathcal{S}$—in particular the intersection curves may end in the middle of $\mathcal{S}$ if $\mathcal{U}$ is not chosen to be properly "aligned" with $\mathcal{S}$.

To alleviate this problem, we use the intersection index to define the primary intersection of the stable and unstable manifolds of a and b, so that the connected intersection curves are submanifolds:

Definition 3 (Primary Intersection) *Let a and b be hyperbolic fixed points of type* A *and type* B*, respectively, whose two dimensional manifolds intersect. We define the primary intersection of the stable and unstable manifolds as*

$$\mathcal{P}(a,b) = \bigcup \{\mathcal{U} \cap \mathcal{S} : \mathcal{U} \in \mathcal{F}^u(b) \; , \; \mathcal{S} \in \mathcal{F}^s(a) \; , \; \kappa(\mathcal{U}, \mathcal{S}) = 0\} \; .$$

We assert that $\mathcal{P}$ is invariant, and is the union of immersed submanifolds of $W^s(a)$ and of $W^u(b)$. Moreover, the intersection of $\mathcal{P}$ with any fundamental domain is generically a *neat submanifold*. Recall that when M is a manifold with boundary, a set $A \subset M$ is neat in M if

$$\partial A = A \cap \partial M \; , \tag{4}$$

(cf. [28] for the definition). In other words, the boundary of the submanifold is *nicely placed* in the boundary of the manifold.

For any fixed fundamental domain $\mathcal{S}$, the primary intersection does not have to be a neat submanifold of $\mathcal{S}$. However, if the intersection of the stable and unstable manifolds in question is transverse, then it always possible to chose a fundamental domain for which the primary intersection is a neat submanifold:

Theorem 2 *Suppose f, a and b are chosen as above, and assume that the primary intersections $P(a,b)$ are transverse. Then*

a) *$\mathcal{P}(a,b)$ is the union of immersed one dimensional manifolds, invariant under f and contained in $W^s(a) \cap W^u(b)$.*

b) *For the generic $\mathcal{U} \in \mathcal{F}^u(b)$, $\bar{\mathcal{U}} \cap \mathcal{P}(a,b)$ is a neat submanifold of $\bar{\mathcal{U}}$.*

c) *For the generic $\mathcal{S} \in \mathcal{F}^s(a)$, $\bar{\mathcal{S}} \cap \mathcal{P}(a,b)$ is a neat submanifold of $\bar{\mathcal{S}}$.*

Proof: Both stable and unstable manifolds are immersed two dimensional manifolds. Since they intersect transversely by assumption, their intersection is the union of one dimensional immersed manifolds. The primary intersection is a subset of this immersed manifold. For each point ξ in the primary intersection we can find a pair of fundamental domains $\mathcal{U}$ and S such that $\xi \in \mathcal{U} \cap \mathcal{S}$. Since the intersection is transverse, ξ is contained in a one dimensional manifold.

If the intersection of $\mathcal{P}$ with a fundamental domain $\mathcal{U}$ is not neat, then since this intersection consists of a union of one dimensional manifolds with boundary, the only possibility is that there is at least one point $\xi \in \partial\{\mathcal{P} \cap \mathcal{U}\}$ that is an interior point of the fundamental domain. By definition there is a fundamental domain $\mathcal{S}_0$ such that $\xi \in \mathcal{U} \cap \mathcal{S}_0$. Then $\xi \in \partial \mathcal{S}_0$, since otherwise, the intersection $\mathcal{U} \cap \mathcal{S}_0 \subset \mathcal{P} \cap \mathcal{U}$ would contain ξ in its interior. However, by continuity, there is a fundamental domain $\mathcal{S}_1$ near $\mathcal{S}_0$, such that $\kappa(\mathcal{U} \cap \mathcal{S}_1) = 0$, and such that $\mathcal{S}_1$ contains ξ in its interior. Thus ξ cannot be on the boundary of the intersection $\mathcal{P} \cap \mathcal{U}$, and so the only possibility is that the boundary of the intersection is contained in the boundary of the fundamental domain. ■

Our definition of primary intersection can be easily generalized to higher dimensions.

4 Melnikov Method for volume preserving maps.

We will use Melnikov's method to show that a perturbation of a degenerate heteroclinic connection between codimension one manifolds typically leads to transverse intersections of stable and unstable manifolds. This approach will help us to study the topology of the primary intersection.

Let $F_0 : \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism preserving the volume (1), such that a and b are hyperbolic fixed points of types A and B respectively. We assume there exists a saddle connection between the fixed points, i.e., $W^s(a) \setminus \{a\} = W^u(b) \setminus \{b\}$. An example of such a map is given below in §5.

We would like to show that after a small perturbation, the manifolds still intersect, as in the classical Melnikov method, but that this intersection is generically transverse and along one dimensional curves. Generally the perturbed map takes the form

$$F_\epsilon = F_0 + \epsilon P_1 ,$$

such that F_ϵ is volume preserving. We make the simplifying assumption that $P_1(a) = P_1(b) = 0$, so that F_ϵ still has hyperbolic fixed points at a and b. However, stated in terms of P_1, it is not so easy to construct volume preserving perturbations to F_0. It is easier to let

$$F_\epsilon = (I + \epsilon P) \circ F_0 ,$$

where I is the identity map. This can always be done since $P \equiv P_1 \circ F_0^{-1}$. In these terms it is easier to construct perturbations that do not destroy the volume preserving property:

Lemma 3 *Let $F_0 : \mathbb{R}^3 \to \mathbb{R}^3$ be a volume preserving diffeomorphism. Then $F_\epsilon = (I + \epsilon P) \circ F_0$ is volume preserving for all ϵ if and only if the Jacobian matrix P' is nilpotent.*

Proof: It is enough to show that $I + \epsilon P$ is volume preserving if and only if $(P')^3 = 0$. This is clear since $\det(I + \epsilon P') = 1$ for all ϵ if and only if the characteristic polynomial of P' is λ^3. ■

The lemma above allows us to easily create examples of perturbations, because if we take any function P such that $(P')^3 \equiv 0$ then the perturbed map is volume preserving. Simple examples include $P(x, y, z) = (0, f(x), g(x, y))$ for any smooth functions f and g that vanish at the fixed points.

Adapted Vector Fields

After perturbation, $W^s(a)$ and $W^u(b)$ will not in general coincide, but will generically intersect transversely. We want to measure the evolution of this intersection as ϵ increases. To do this we need to define a pair of independent and invariant vector fields on the manifolds. We call such vector fields "adapted":

Definition 4 (Adapted vector field) *A vector field V on the saddle connection $W^s(a) \cap W^u(b)$ is said to be* adapted *to the dynamics of F_0 if*

a) $(F_0)_* V = V$,

b) $\lim_{\xi \to a} V(\xi) = 0$,

c) $\lim_{\xi \to b} V(\xi) = 0$.

Recall that, if $G : M \to N$ is a diffeomorphism and $V \in TM$ is a vector field then $G_* V = G'(G^{-1}(\xi))V(G^{-1}(\xi))$ is a vector field on TN. Thus the condition $(F_0)_* V = V$ is equivalent to $F'(\xi)V(\xi) = V(F_0(\xi))$ for all ξ in the saddle connection. Also, if V is continuous on $W^s(a) \cap W^u(b) \cup \{a, b\}$ the first condition implies the other two. It is easy to see that adapted vector fields always exist when the multipliers of the fixed points are complex:

Lemma 4 *Assume F_0 has a saddle connection as defined as above, and that the Jacobians $F_0'(a)$ and $F_0'(b)$ have pairs of complex conjugate multipliers that define their stable and unstable subspaces, respectively. Then there exist two adapted vector fields V, W defined on the saddle connection that are linearly independent for all points in the saddle connection apart from at a and b*

Proof: The Stable Manifold theorem [29] implies that there exists a diffeomorphism $\phi : \mathbb{R}^2 \to W^s(a)$ such that $\phi(0) = a$ and

$$\phi \circ L \circ \phi^{-1} = F_0 \ ,$$

where $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map $L(x) = Ax$ and A is the derivative of F_0 restricted to its stable plane. Thus, by assumption A has two complex eigenvalues with equal norm less than 1. It is enough to show that there are two linearly independent vector fields $\tilde{V}, \tilde{W}$ in $\mathbb{R}^2$ such that $\tilde{V}(Ax) = A\tilde{V}(x)$ and $\tilde{W}(Ax) = A\tilde{W}(x)$, for all $x \in \mathbb{R}^2$. We can choose $\tilde{V}(x) = x$ and $\tilde{W}(x) = Ax$. Then $V = \phi_*\tilde{V}$ and $W = \phi_*\tilde{W}$ satisfy the first two conditions of Defn. 4.

To show that the last property is satisfied, note that we can construct a similar diffeomorphism for the unstable manifold: $\psi : \mathbb{R}^2 \to W^u(b)$. Then DF_0 is conjugate to a matrix B which has two complex eigenvalues of equal norm greater than 1. The corresponding vector fields x and Bx are linearly independent, and their push forward onto $W^u(b)$ satisfies the first and third conditions of Defn. 4. Moreover, at every point except the two fixed points, these vector fields must be a linear combination of V and W. Thus V and W satisfy condition (c). ■

In more general cases, it can also be shown, albeit with more effort, that adapted vector fields exist, however, the case covered by lemma 4 is sufficient for the examples that we study in this paper.

Melnikov Function

Based on the vector fields found above, we will define a global Melnikov function $M(\xi)$ on the saddle connection that is invariant under F_0.

Definition 5 (Melnikov function) *Let $F_\epsilon = F_0 + \epsilon P \circ F_0$, be a volume preserving map with type A and type B fixed points at a and b respectively, such that F_0 has a two dimensional saddle connection between a and b. Suppose that V, W are a pair of adapted vector fields that are linearly independent on the saddle connection. For any $\xi \in W^s(a) \cap W^u(b)$ the Melnikov function is defined as*

$$M(\xi) = \sum_{k=-\infty}^{\infty} \det(P(\xi_k), V(\xi_k), W(\xi_k)) \ , \tag{5}$$

where $\xi_k = F_0^k(\xi)$.

As we will show below, M measures the distance between the perturbed manifolds. In order that it be useful, M should in some sense be independent of the choice of adapted vector fields:

Proposition 5 *The set of zeros of M is independent of the choice of the vector fields V, W and is invariant under F_0. This is also true for the nondegenerate zeros.*

Proof: Let V, W and $\tilde{V}, \tilde{W}$ be two pairs of independent, adapted vector fields. Let M be the Melnikov function defined using V and W and $\tilde{M}$ be the Melnikov function defined using $\tilde{V}$ and $\tilde{W}$. Since each pair is linearly independent, it is possible to find functions α, β, γ and δ such that

$$\begin{aligned} \tilde{V} &= \alpha V + \beta W \ , \\ \tilde{W} &= \gamma V + \delta W \ . \end{aligned}$$

Since both pairs of vector fields are independent, it is clear that $d = \alpha\delta - \beta\gamma \neq 0$. In addition, $\tilde{V}(\xi) \times \tilde{W}(\xi) = d(\xi)(V(\xi) \times W(\xi))$. Using (2), (3), and the invariance of the volume form we see that

$$\begin{aligned} d(\xi)^2 &= \frac{\det(\tilde{V}(\xi), \tilde{W}(\xi), \tilde{V}(\xi) \times \tilde{W}(\xi))}{\det(V(\xi), W(\xi), V(\xi) \times W(\xi))} \\ &= \frac{\det(F_0'(\xi)\tilde{V}(\xi), F_0'(\xi)\tilde{W}(\xi), F_0'(\xi)(\tilde{V}(\xi) \times \tilde{W}(\xi)))}{\det(F_0'(\xi)V(\xi), F_0'(\xi)W(\xi), F_0'(\xi)(V(\xi) \times W(\xi)))} \\ &= \frac{\det(\tilde{V}(F_0(\xi)), \tilde{W}(F_0(\xi)), F_0'(\xi)(\tilde{V}(\xi) \times \tilde{W}(\xi)))}{\det(V(F_0(\xi)), W(F_0(\xi)), F_0'(\xi)(V(\xi) \times W(\xi)))} \\ &= d(F_0(\xi))d(\xi) \end{aligned}$$

Therefore $d(F_0(\xi)) = d(\xi)$ and from this we conclude that

$$\tilde{M}(\xi) = d(\xi)M(\xi)\ . \tag{6}$$

Using (6), we conclude that ξ^* is a zero for M if and only if it is a zero for $\tilde{M}$. Moreover, since $d > 0$ a zero of M is nondegenerate if and only if it is nondegenerate for $\tilde{M}$ as well. ■

Therefore if we are only interested in the set of zeros of M, this proposition allows us to make a local analysis of the Melnikov function in order to find these zeros. Once we have the set of zeros for a fundamental domain, we find the rest by iteration. In other words, we can restrict our analysis to fundamental domains. Moreover, the nondegenerate zeros of M define the primary intersections of the manifolds:

Theorem 6 *Let M be a Melnikov function as in (5).*

a) *If ξ^* is a nondegenerate zero of M, then $W^u(b, F_\epsilon)$ and $W^s(a, F_\epsilon)$ intersect transversely near ξ^*, for ϵ small enough.*

b) *The set of nondegenerate zeros can continued, for ϵ small enough, to the primary intersection of $W^u(b, F_\epsilon)$ and $W^s(a, F_\epsilon)$.*

We give the proof in the appendix.

Melnikov function when there is an integral

Computing the Melnikov function defined in (5) could be difficult if one needs to construct a pair of adapted vector fields explicitly. However, if F_0 has a first integral I, then we can use it to simplify the computations. In fact, we only need to assume that there is a local integral for F_0 in the sense that in some neighborhood of the saddle connection there is an invariant. Restricting the volume form to a surface of constant I gives an invariant area:

Lemma 7 *Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be a volume preserving diffeomorphism and M a surface that is given locally as the zero set of a function I. Assume that I is invariant under G in some* *neighborhood of M. Then the set $\mathcal{H} = \{\xi : \nabla I(\xi) \neq 0\}$ is invariant and G preserves the 2-form*

$$\omega(\xi)(v, w) = \det\left(\frac{\nabla I(\xi)}{|\nabla I(\xi)|^2}, v, w\right) \tag{7}$$

on each level set $\{I = k\} \cap \mathcal{H}$.

Proof: Since I is an integral of G near M, $I \circ G = I$, and therefore $G'(\xi)^T \nabla I(G(\xi)) = \nabla I(\xi)$. Let $T_\xi(M)^\perp = \operatorname{span}\{\nabla I(\xi)\}$. We are interested in finding the projection of $G'(\xi)\nabla I(\xi)$ onto the normal space $T_{G(\xi)}(M)^\perp = \operatorname{span}\{\nabla I(G(\xi))\}$. In order to do this, it is enough to find the dot product between $G'(\xi)\nabla I(\xi)$ and $\nabla I(G(\xi))$:

$$G'(\xi)\nabla I(\xi) \cdot \nabla I(G(\xi)) = \nabla I(\xi) \cdot G'(\xi)^T \nabla I(G(\xi)) = |\nabla I(\xi)|^2 \, .$$

The pull back of ω by G is

$$\begin{aligned}
G^*\omega(\xi)(v, w) &\equiv \omega(G(\xi))(G'(\xi)v, G'(\xi)w) \\
&= \det\left(\frac{\nabla I(G(\xi))}{|\nabla I(G(\xi))|^2}, G'(\xi)v, G'(\xi)w\right) \\
&= \det\left(\frac{G'(\xi)\nabla I(\xi)}{|\nabla I(\xi)|^2}, G'(\xi)v, G'(\xi)w\right) \\
&= \det\left(\frac{\nabla I(\xi)}{|\nabla I(\xi)|^2}, v, w\right) \\
&= \omega(\xi)(v, w) \, .
\end{aligned}$$

■

If we let $G = F_0$ then the saddle connection that we are using for the Melnikov method satisfies the conditions of lemma 7. This allows us to rewrite the formula for the Melnikov function in terms of ∇I:

Lemma 8 *Let I be an integral for F_0, such that the surface $I = k$ is a saddle connection. Then it is possible to choose the vector fields V and W in (5) so that*

$$M(\xi) = \sum_{k=-\infty}^{\infty} < P(\xi_k), \nabla I(\xi_k) > \tag{8}$$

where $\xi_k = F_0^k(\xi)$.

Proof: Let $\tilde{V}$ and $\tilde{W}$ be any pair of adapted vector fields that are linearly independent on the saddle connection. Since $\tilde{V}, \tilde{W} \in T_\xi M$, then there exists a nonzero function d such that, for every ξ on the saddle connection

$$\tilde{V}(\xi) \times \tilde{W}(\xi) = d(\xi)\nabla I(\xi) \, .$$

We conclude that $d(\xi) = \omega(\xi)(\tilde{V}(\xi), \tilde{W}(\xi))$, where ω is defined as (7). We see that d is invariant since

$$\begin{aligned} d(F_0(\xi)) &= \omega(F_0(\xi))(\tilde{V}(F_0(\xi)), \tilde{W}(F_0(\xi))) \\ &= \omega(F_0(\xi))(F_0'(\xi)\tilde{V}(\xi), F_0'(\xi)\tilde{W}(\xi)) \\ &= F_0^*\omega(\xi)(\tilde{V}(\xi), \tilde{W}(\xi)) \\ &= \omega(\xi)(\tilde{V}(\xi), \tilde{W}(\xi)) = d(\xi) \ . \end{aligned}$$

If we let $V = \tilde{V}$ and $W = \tilde{W}/d$, then

$$\det(P(\xi), V(\xi), W(\xi)) =< P(\xi), \nabla I(\xi) >$$

and this implies what we want. ■

5 Examples

In this section we construct a family of volume preserving maps that have a completely degenerate heteroclinic intersection (i.e., a saddle connection). We obtain this family as a semidirect product of an area preserving, twist map and a rotation in three space. The twist map is defined in such a way that it has two invariant sets which give rise to a saddle connection between two fixed points. Examples similar to these were found by Lomelí [22] and are related to the work of Suris [30, 31] on integrable maps. It is interesting to note the map need not have an integral, and therefore, apart from the two invariant sets, typically exhibits chaotic behavior. We finally give an example for which the resulting volume preserving map has a first integral.

Explicit Heteroclinic Connection

We start with the area preserving map generated by the Lagrangian generating function $L : \mathbb{R}^2 \to \mathbb{R}$,

$$L(z, Z) = -zZ + \int_0^z h(\zeta)d\zeta + \int_0^Z h^{-1}(\zeta)d\zeta \ , \tag{9}$$

where $h : \mathbb{R} \to \mathbb{R}$ is any strictly increasing (i.e., $h' > 0$), C^2 diffeomorphism. The Lagrangian generates a *twist* map $f(r, z) = (R, Z)$ that is implicitly defined by

$$dL = RdZ - rdz \ .$$

The map is automatically area preserving since $0 = d^2L = dR \wedge dZ - dr \wedge dz$. Explicitly, we obtain

$$f(r, z) = \begin{pmatrix} h^{-1}(r + h(z)) - z \\ r + h(z) \end{pmatrix} \ . \tag{10}$$

It is easy to verify that the map has two invariant manifolds, the z-axis and the curve

$$\mathcal{C} = \{(r, z) : r = h^{-1}(z) - h(z)\} \ .$$

If h has a fixed point z^*, then the invariant curves intersect at the point $(0, z^*)$ which is a fixed point of f. The linearization of f at such a fixed point is

$$f'(0, z^*) = \begin{pmatrix} 1/h'(z^*) & 0 \\ 1 & h'(z^*) \end{pmatrix},$$

so that the fixed point is hyperbolic whenever $h'(z^*) \neq 1$. For example, when $h'(z^*) > 1$ the z-axis is the unstable manifold, and $\mathcal{C}$ is the stable manifold. The stabilities are exchanged when $h'(z^*) < 1$. Thus if h has two neighboring isolated fixed points, the invariant curves provide heteroclinic connections between them; see the sketch in Fig. 3.

We can extend this twist map to $\mathbb{R}^3$ by introducing the cylindrical angle θ, and letting $\sqrt{2r}$ be the cylindrical radius. Equivalently, the rectangular coordinates

$$\begin{aligned} x &= \sqrt{2r}\cos\theta \\ y &= \sqrt{2r}\sin\theta, \end{aligned}$$

are defined so that $dx \wedge dy = dr \wedge d\theta$. The map f then extends to a map $f_0 : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$f_0(x, y, z) = \begin{pmatrix} \rho(r, z)x \\ \rho(r, z)y \\ r + h(z) \end{pmatrix},$$

where $r = \frac{1}{2}(x^2 + y^2)$, and

$$\rho(r, z) = \begin{cases} \sqrt{\dfrac{h^{-1}(r + h(z)) - z}{r}}, & r \neq 0 \\[2ex] \dfrac{1}{\sqrt{h'(z)}}, & r = 0. \end{cases} \tag{11}$$

The map f_0 is as smooth as the diffeomorphism h:

Lemma 9 *Assume that $h \in C^{r+1}(\mathbb{R})$ with $r > 1$, and ρ is defined by (11). Then $\rho \in C^r(\mathbb{R}^3)$. In addition, if h is analytic, so is ρ.*

Proof: Let $\zeta(r, z) = \int_0^1 (h^{-1})'(rs + h(z))ds$. It is easy to see that $\zeta \in C^r(\mathbb{R}^3)$ and that $\rho(r, z) = \sqrt{\zeta(r, z)}$. In addition since h is assumed to be strictly increasing, then $\zeta > 0$, which implies what we want. ■

The map becomes fully three dimensional if we introduce dynamics in θ. To do this, we compose the map with a rotation about the $z - axis$. Denote such a rotation by angle α by

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

Since the map f_0 is rotationally invariant, it can be composed with a rotation whose angle $\alpha = 2\pi\omega(r, z)$ depends smoothly on (r, z) to define a diffeomorphism $F_0(x, y, z) = (X, Y, Z)$ by

$$F_0 = f_0 \circ R_{2\pi\omega}, \tag{13}$$

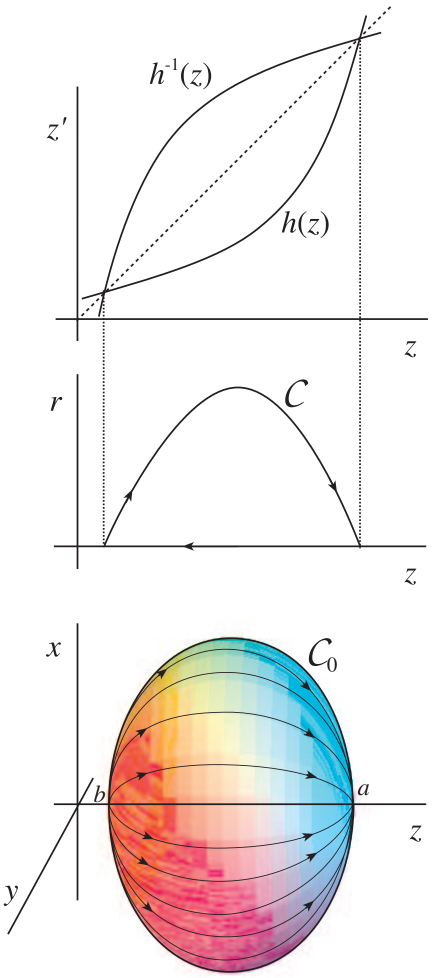

Figure 3: Construction of the invariant curves $r = 0$ and $\mathcal{C}$ for an h with two fixed points, and the manifold $\mathcal{C}_0$ for the volume preserving map.

with the desired properties.

First, F_0 preserves the volume form (1) because

$$\begin{aligned} F_0^*\Omega &= dX \wedge dY \wedge dZ = dZ \wedge dR \wedge d\Theta \\ &= dz \wedge dr \wedge (d\theta + 2\pi\omega_r dr + 2\pi\omega_z dz) \\ &= dr \wedge d\theta \wedge dz = \Omega . \end{aligned}$$

Moreover, the z-axis and the surface

$$\mathcal{C}_0 = \left\{ \frac{1}{2}(x^2+y^2) = h^{-1}(z) - h(z) \right\}$$

are invariant. These two manifolds intersect at points $(0, 0, z^*)$ for which $h(z^*) = z^*$—these are fixed points for F_0. The derivative of F_0 at such a fixed point is

$$F_0'(0,0,z^*) = \begin{pmatrix} \rho^* \cos 2\pi\omega^* & -\rho^* \sin 2\pi\omega^* & 0 \\ \rho^* \sin 2\pi\omega^* & \rho^* \cos 2\pi\omega^* & 0 \\ 0 & 0 & 1/\rho^{*2} \end{pmatrix} \tag{14}$$

where $\rho^* \equiv \rho(0, z^*) = 1/\sqrt{h'(z^*)}$ and $\omega^* = \omega(0, z^*)$. Thus if $h'(z^*) > 1$ (< 1) the fixed point is type A (B), with stable (unstable) manifold given by $\mathcal{C}_0$, and unstable (stable) manifold given by the z-axis.

Finally, the manifold $\mathcal{C}_0$ is a two dimensional heteroclinic connection for two neighboring fixed points of h. This situation is illustrated in Fig. 3. Since the multipliers on the two dimensional manifolds are complex, the diffeomorphism F_0 has a pair of adapted vector fields as shown in lemma 4.

Because the map F_0 is a semi-direct product of a rotation about the z-axis with the map (10), it commutes with rotations about the z axis. That is it has the symmetry

$$F_0 \circ R_\alpha = R_\alpha \circ F_0 \ . \tag{15}$$

where the rotation is given by (12).

If we assume that ω is constant, then we can give an explicit formula for all the iterates of F_0 on the saddle connection $\mathcal{C}_0$, in terms of the iterates of h:

Lemma 10 *Suppose that ω is a constant. Let $\xi = (x_0, y_0, z_0) \in \mathcal{C}_0$, and $x_0 = \sqrt{2r_0}\cos\theta_0$ and $y_0 = \sqrt{2r_0}\sin\theta_0$, where $r_0 = h^{-1}(z_0) - h(z_0)$. Then, for all $k \in \mathbb{Z}$, the k^{th} iterate of F_0 is*

$$F_0^k(\xi) = \begin{pmatrix} \sqrt{2r_k}\cos(\theta_0 + 2\pi k\omega) \\ \sqrt{2r_k}\sin(\theta_0 + 2\pi k\omega) \\ h^{-k}(z_0) \end{pmatrix}$$

where $r_k = h^{-k-1}(z_0) - h^{-k+1}(z_0)$.

Integrable Volume Preserving Map

In order to compute the Melnikov function M of (5), it is advantageous to choose h so that its iterates can be evaluated explicitly. In addition, it is desirable to have a first integral for the map F_0 to simplify the Melnikov function, as in lemma 7. One such choice is (c.f. [30]):

Lemma 11 *Let $h : \mathbb{R} \to \mathbb{R}$ be the diffeomorphism*

$$h_\nu(z) = z - \frac{2}{\pi} \arctan\left(\frac{(1-\nu)\cos\pi z}{(1+\nu)+(1-\nu)\sin\pi z}\right) , \tag{16}$$

where $0 < \nu < 1$. Then

- **a)** $h_\nu^t = h_{\nu^t}$, *for all $t \in \mathbb{Z}$.*
- **b)** $h_\nu(-z) = -h_{\nu^{-1}}(z)$.
- **c)** *On the subinterval $(-1/2, 1/2)$, the diffeomorphism h_ν is conjugate to a Möbius transformation. In other words, we can rewrite*

$$h_\nu(z) = \frac{2}{\pi} \arctan\left(T_\nu(\tan(\frac{\pi}{2}z))\right) ,$$

where

$$T_\nu(w) = \frac{(\nu+1)w + (\nu-1)}{(\nu-1)w + (\nu+1)} .$$

- **d)** $-\frac{1}{2}$ *is a stable fixed point and* $\frac{1}{2}$ *is an unstable fixed point of h_ν.*

With this choice of h, the point $a = (0, 0, 1/2)$ is a type A, and $b = (0, 0, -1/2)$ is a type B, fixed point for $F_0(x, y, z)$.

In this case the twist map (10) generated by the diffeomorphism (16) has the first integral

$$J(r, z) = 2\nu\cos(\pi r) + (1-\nu^2)\cos(\pi z)\sin(\pi r) .$$

Some of the levels sets of J are shown in Fig. 4. Since F_0 is obtained from the area preserving map by a symplectic rotation about $r = 0$, the function

$$I(x, y, z) = J\left(\frac{1}{2}(x^2 + y^2), z\right) \tag{17}$$

is an invariant for F_0.

The symmetry (b) in Lemma 11 implies that the two dimensional map is reversible,

$$f \circ S = S \circ f^{-1} \quad \text{where } S(r, z) = (r, -z) . \tag{18}$$

For the case that ω is constant, this implies that F_0 has the reversor

$$S_0(x, y, z) = (x, -y, -z) .$$

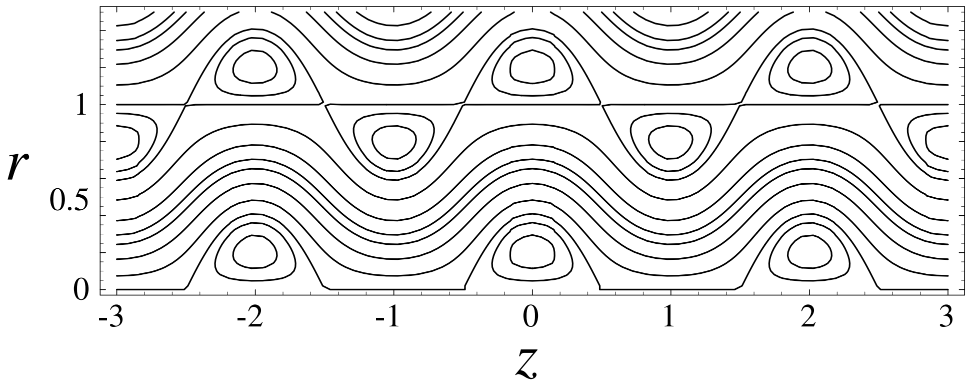

Figure 4: Contours of J when $\nu = 0.5$.

To see this, note that both the rotation, $S_0 R_{2\pi\omega} = R_{-2\pi\omega} S_0$, and f_0 are reversed by S_0. Moreover, the rotation commutes with f_0 when ω is constant; therefore,

$$\begin{aligned} F_0 \circ S_0 &= f_0 \circ R_{2\pi\omega} \circ S_0 = f_0 \circ S_0 \circ R_{-2\pi\omega} \\ &= S_0 \circ f_0^{-1} \circ R_{-2\pi\omega} = S_0 \circ F_0^{-1} \ . \end{aligned}$$

The fixed line of this reversor is the x-axis, and $S_0(a) = b$. A standard argument [32] implies that points where $W^s(a)$ crosses the x-axis are heteroclinic to b.

Lemma 7 implies that the invariant I can be used to simplify the computation of the Melnikov function. Recall that the choice of perturbation P should be zero at the fixed points and be such that P' is nilpotent (c.f. Lemma 3). One such choice is

$$P(x, y, z) = (0, \mu x^2/2, (1-\mu)(x^2+y^2)/2) \ . \tag{19}$$

Since $(I + \epsilon P)$ is reversed by the operator S_0, we see that when ω is constant

$$F_\epsilon \circ S_0 = S_0 \circ F_\epsilon^{-1} \ .$$

Thus, there are always heteroclinic points where W^s crosses the x-axis.

Using lemma 10, the corresponding Melnikov function M for F_ϵ is given by

$$M(\xi) = \sum_{k=-\infty}^{\infty} \Lambda(F_0^k(\xi)) \ ,$$

where $\Lambda(x, y, z) =< P(x, y, z), \nabla I(x, y, z) >$.

For the case that ω is constant, we have $F_0^k(S_0\xi) = S_0 \circ F_0^{-k}(\xi)$, so that

$$M(S_0\xi) = \sum_{k=-\infty}^{\infty} \Lambda(S_0 \circ F_0^{-k}(\xi)) \ .$$

Moreover, $\Lambda(S_0\xi) = -\Lambda(\xi)$, for the perturbation (19). Thus,

$$M(x, -y, -z) = -M(x, y, z) \ , \tag{20}$$

which implies in particular that $M(x, 0, 0) = 0$.

6 Bifurcation of primary intersections

In this section, we compute the Melnikov function for the map (13) with the diffeomorphism, h given by (16), and the perturbation (19). We will see that the topology of the heteroclinic intersection changes as the parameters μ, ν and ω of P and F_0 vary.

We consider the simplest case where ω is a constant. This implies that the local motion on the stable and unstable manifolds of the fixed points is a spiral with rotation number ω. Recall that the heteroclinic connection is the topological sphere defined by $\mathcal{C}_0 = \{(r,z) : r = h^{-1}(z) - h(z)\}$. Note that by (13) that $Z = r + h(z)$ depends only upon r and z, and not upon the spherical angle θ. Thus the equator is a proper loop, γ. Using the notation

$$H(\nu) = -h_\nu(0) = \frac{2}{\pi}\arctan\left(\frac{1-\nu}{1+\nu}\right) ,$$

then the equator is the circle $\gamma = \{(x,y,z) : z = 0, r = 2H(\nu)\}$, and its iterate is $F_0(\gamma) = \{(x,y,z) : z = H(\nu), r = H(\nu^2)\}$. Thus a fundamental domain on $W^s(a)$ is the annulus defined by the interval $0 \le z < H(\nu)$.

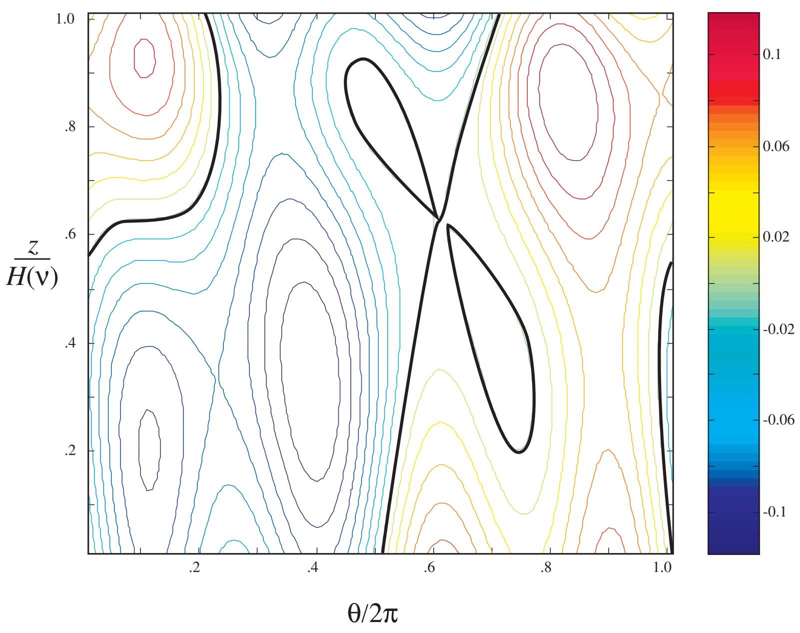

Figure 5: Contours of the Melnikov function for $\nu = 0.2$, $\mu = 0.1$, and $\omega = 0.2$, near the cusp bifurcation in Fig. 6. Here M ranges from -0.12 to 0.12, and the zero level, shown as the black curves, consists of a pair of curves of homotopy class $(1,0)$, and pair of bubbles (homotopy class $(0,0)$) that have nearly collided with one of the $(1,0)$ curves.

We show an example of the Melnikov function in Fig. 5, using the coordinates z and θ on the fundamental domain. Positive values of $M(\theta, z)$ are shown in shades of red, and negative in shades of blue, and the zero level is shown as the solid black curve. As implied by theorems 6 and 2 the contours of M are neat submanifolds of the fundamental annulus—either closed loops or curves that end on one of the boundaries of the annulus.

In general, since the boundaries of the fundamental domain are γ and $F(\gamma)$, we may use the map F to identify the boundaries the annulus, turning it into a torus. In our example F_0 rigidly rotates the equatorial circle by $2\pi\omega$, so that we merely undo this rotation to perform the identification:

$$\{\theta + 2\pi\omega, H(\nu)\} \equiv \{\theta, 0\}$$

Since the zeros of M are neat submanifolds, they become closed loops with this identification. Thus the zero contours of M can be classified by their homotopy class, a pair of integers (m, n) that gives the number of times the contour loops around the torus in the z and θ directions, respectively. When the zeros of M are nondegenerate, all of the curves must have either the same homotopy class, or the class $(0, 0)$. Each loop has a natural direction, associated with the direction of the crossing of the manifolds. Thus loops with a nontrivial homotopy class must appear in pairs.

In Fig. 5 there are a pair of loops of homotopy class $(1, 0)$, i.e., curves that extend from $z = 0$ to $z = H(\nu)$ without encircling longitudinally. By the symmetry (20) there are always zeros on the x-axis, so $M(0, 0) = M(\pi, 0) = 0$—in the case shown the primary intersection curves through these points have homotopy class $(1, 0)$. Also shown in the figure are a pair of loops of homotopy class $(0, 0)$, i.e., loops that are homotopic to a point. These loops appear in a parameter region corresponding to small ω and moderate values of μ, and disappear either by colliding with a $(1, 0)$ loop, or by shrinking to a point. For example, if we fix $\nu = 0.2$, $\mu = 0.1$, then for the range $0 \le \omega < 0.105$ the $(0, 0)$ loops exist. At $\omega \approx 0.105$ the loops shrink to a point, and for $0.105 < \omega < 0.185$, there is a single pair of $(1, 0)$ curves. At $\omega \approx 0.185$ a new pair of loops are born, and these are finally destroyed in a collision with the $(1, 0)$ curve just above $\omega = 0.2$.

A complete picture of the primary intersections for $\nu = 0.2$ is shown in the bifurcation diagram Fig. 6. Here we can restrict the range of ω to the interval $[0, 0.5]$, since a rotation about the z-axis by $2\pi\omega$ is conjugate to one by $2\pi(1-\omega)$ under the coordinate transformation $\theta \mapsto -\theta$. There are four distinct regions in Fig. 6, corresponding to loops with homotopy classes $(0, 1)$, $(1, 0)$, $(3, 1)$, and $(1, 0)$ with a pair of trivial loops. The parameters for Fig. 5 are near the codimension two, cusp point at $(\omega, \mu) \approx (0.2, 0.15)$, which corresponds both to the collision of the trivial loops with a $(1, 0)$ loop, and their shrinking to a point. Examples of the zero contours of the Melnikov function are shown in Fig. 7, corresponding to the parameter values labeled (a)-(f) in Fig. 6. When μ is small, the intersection curves are "equatorial", of class $(0, 1)$; this corresponds to panel (d) in Fig. 7. For small ω and moderate values of ν the primary intersections correspond to a pair of $(1, 0)$ curves plus a pair of "bubbles", curves with homology class $(0, 0)$, as shown panel (a) of Fig. 7. As ω increases these bubbles disappear, leaving only the $(1, 0)$ curves, shown in panel (b),(e) and (f) of Fig. 7. These become increasingly elongated as one approaches the $(3, 1)$ bifurcation where they reconnect, as shown in panel (c) of Fig. 7, forming a single pair of $(3, 1)$ loops.

To compare the actual behavior of the manifolds for the map F_ϵ, we need to choose a reasonably large value of ϵ so that the intersections can be numerically resolved. It is

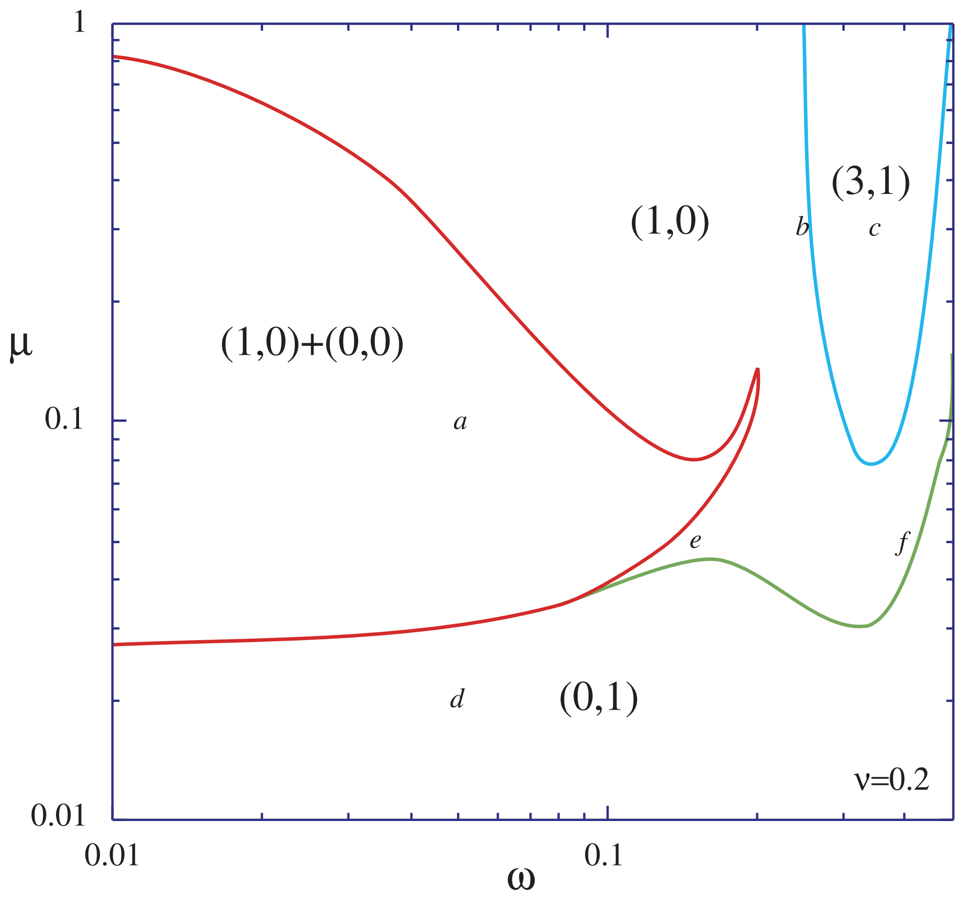

Figure 6: Bifurcation Diagram for the Melnikov function when $\nu = 0.2$ as a function of the rotation number ω of the fixed points and the parameter μ of the perturbation P. The points labeled (a)-(f) correspond to the parameter values in Fig. 7 and Fig. 8.

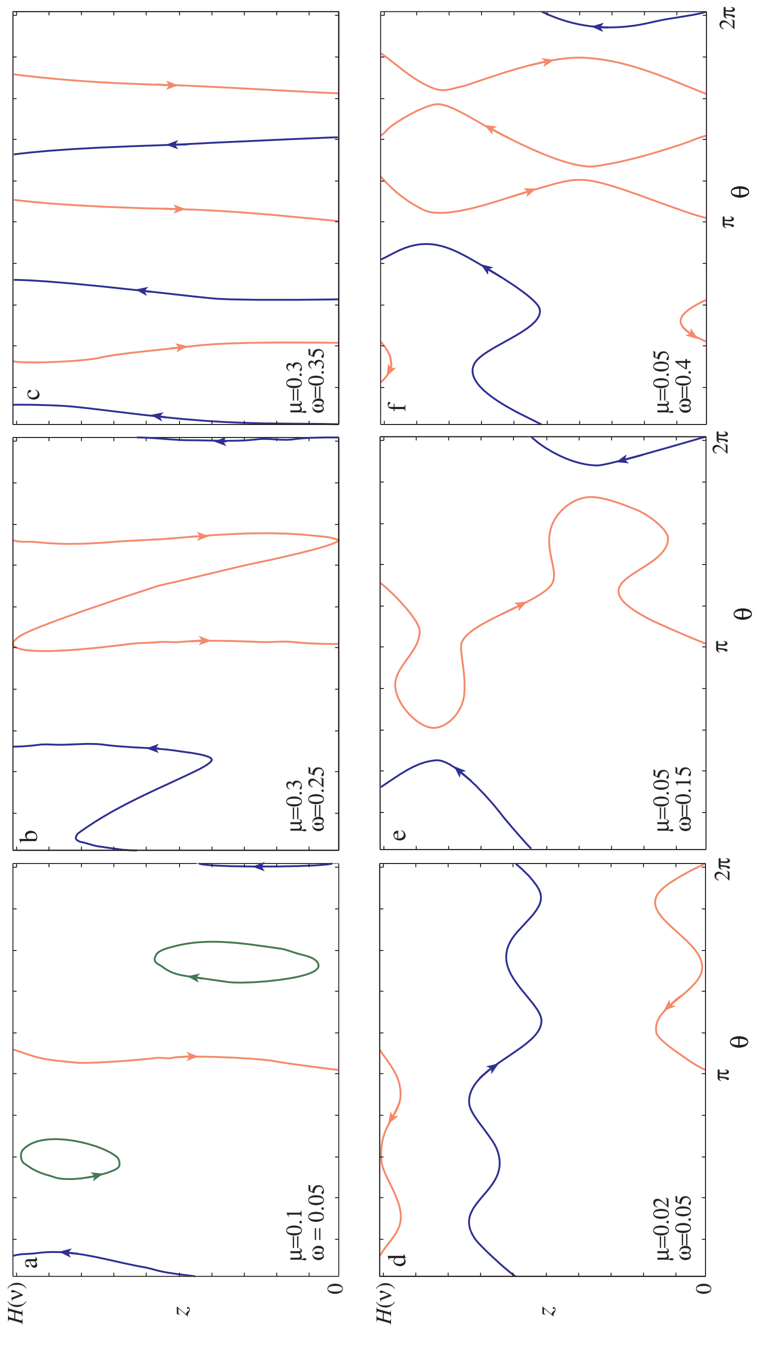

Figure 7: Zero contours of the Melnikov function for $\nu = 0.2$, and parameter values (a)-(f) labeled in Fig. 6.

relatively easy to plot the manifold $W^s(a)$ when the pair of stable multipliers at the fixed point have the same magnitude [25]; this is true for our map by (14). In this case one can take a regular two dimensional grid whose size is order unity, and create a grid adapted to the dynamics by iterating the points with the linearization of the map restricted to the stable subspace N times. This "small" grid is now embedded into the tangent plane of $W^s(a)$ at a and iterated N steps with the inverse of the fully nonlinear map. The resulting grid now approximately falls along the stable manifold, and is roughly regularly spaced. A similar algorithm can be used for the unstable manifold of b.

In Fig. 8 we show three dimensional pictures of the manifolds created with this algorithm for $\epsilon = 0.75$ and the same six values of (ω, μ) in Fig. 7. All of the intersections have the same homotopy types as the predictions with the exception of panel (f), at $(\omega, \mu) = (0.4, 0.05)$, for which the Melnikov function predicts $(1, 0)$, and the actual intersections in the numerical picture appear to be $(0, 1)$. This is due to the fact that the parameters are close to the $(1, 0) \leftrightarrow (0, 1)$ bifurcation curve, and that ϵ is quite large.

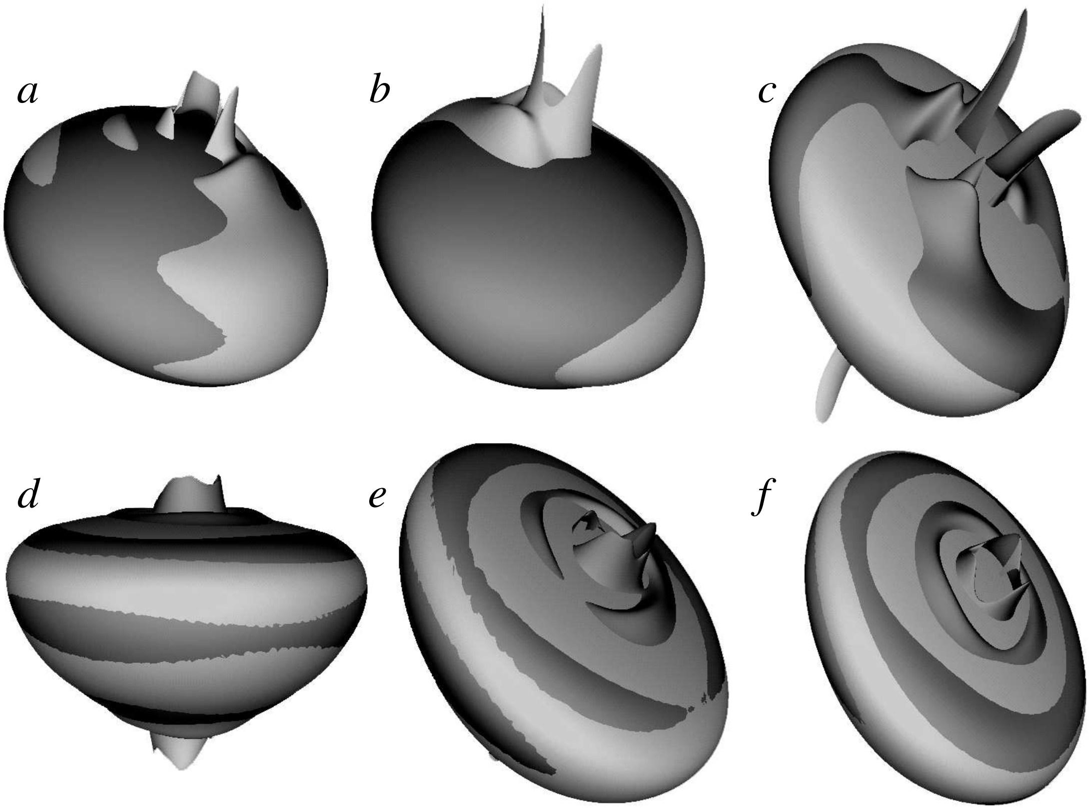

Figure 8: Stable and Unstable manifolds for F_ϵ with $\epsilon = 0.75$. Here the parameters (ω, μ) in each panel are identical to those in Fig. 7. $W^u(b)$ is shown as dark gray, and the $W^s(a)$ as light gray.

7 Conclusions

We have generalized the definitions of fundamental domains and primary intersections to $\mathbb{R}^3$ and provided and some tools for their study. In particular, a codimension one Melnikov method has been used to identify primary intersections between two dimensional stable and unstable manifolds in a family of volume preserving maps.

The heteroclinic intersections, which are generically curves, can be labeled by their homotopy class. We have shown that there are bifurcations between these classes, and that which occurs will depend, for example, on the complex phase of the multiplier of the associated fixed point. Heteroclinic orbits can be found most easily for the reversible case, as intersections should occur on the fixed set of the reversor. In our example the reversor has a fixed line, the x-axis.

One of our motivations for characterizing volume preserving maps is to study transport. If the two dimensional manifolds intersect on an equatorial circle, then transport can be localized to "lobes" similar to the two dimensional case [15]. However, if the primary intersection has a different homotopy class, then the construction of "lobes" entirely from pieces of stable and unstable manifold may be impossible.

Appendix: Proof of Theorem 6

For each point ξ in the saddle connection $W^s(a) \cap W^u(b)$, there is a neighborhood $\mathcal{N}_0$ contained in a fundamental domain of the saddle connection, such that all the iterates $f^k(\mathcal{N}_0)$ are disjoint. On the other hand, there is an $\varepsilon_0 > 0$ and a smooth function $\phi : \mathcal{N}_0 \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^3$ such that

a) $\forall \varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\phi(\xi, \varepsilon) \in W^u(b, F_\varepsilon)$,

b) $\phi(\xi, 0) = \xi$.

Let $\mathcal{V} = \bigcup_{k=0}^{\infty} F_0^k(\mathcal{N}_0)$. Clearly $\mathcal{V}$ is a immersed manifold. Moreover, we can extend the domain of ϕ to all of $\mathcal{V}$, by defining

$$\phi(\xi, \varepsilon) = F_\varepsilon^k(\phi(F_0^{-k}(\xi), \varepsilon)) \;, \tag{21}$$

provided $\xi \in F_0^k(\mathcal{N}_0)$. It is clear that for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $\xi \in \mathcal{V}$, we have that $\phi(\xi, \varepsilon) \in W^u(b, F_\varepsilon)$.

For each ξ, we are interested in estimating $\phi(\xi, \varepsilon)$ to first order in ε. Using (21) with $k = 1$, we can take the partial derivative of ϕ with respect to ε to obtain the relation

$$\partial_\varepsilon \phi(\xi, 0) = F_0'(F_0^{-1}(\xi))\partial_\varepsilon \phi(F_0^{-1}(\xi), 0) + P(\xi) \;. \tag{22}$$

Let V and W is a pair of linearly independent, adapted vector fields (cf. Defn. 4). We observe that the property of being adapted implies that, for all ξ in the saddle connection,

$$\begin{aligned} V(\xi) &= F_0'(F_0^{-1}(\xi)V(F_0^{-1}(\xi)) \;, \\ W(\xi) &= F_0'(F_0^{-1}(\xi)W(F_0^{-1}(\xi)) \;. \end{aligned} \tag{23}$$

The vector $V(\xi)\times W(\xi)$ is normal to the saddle connection, so

$$< \partial_\varepsilon\phi(\xi,0), V(\xi)\times W(\xi) >= \det(\partial_\varepsilon\phi(\xi,0), V(\xi), W(\xi)) \; , \tag{24}$$

is a measure of the normal deviation of the unstable manifold, as ε varies. Now, using (22) and (23), we find

$$\begin{aligned} < \partial_\varepsilon\phi(\xi,0), V(\xi)\times W(\xi) > \;&=\; \det(\partial_\varepsilon\phi(F_0^{-1}(\xi),0), V(F_0^{-1}(\xi)), W(F_0^{-1}(\xi))) \\ &\quad + \det(P(\xi), V(\xi), W(\xi)) \; . \end{aligned}$$

Upon iteration this relation implies that for all integers $n\geq 1$,

$$\begin{aligned} < \partial_\varepsilon\phi(\xi,0), V(\xi)\times W(\xi) > \;&=\; \det(\partial_\varepsilon\phi(F_0^{-n}(\xi),0), V(F_0^{-n}(\xi)), W(F_0^{-n}(\xi))) \\ &\quad + \sum_{k=0}^{n-1}\det(P(F_0^{-k}(\xi)), V(F_0^{-k}(\xi)), W(F_0^{-k}(\xi))) \; . \end{aligned}$$

Notice that $\partial_\varepsilon\phi(\xi,0)$ is bounded near $\xi=b$ and since the vector fields are adapted, $\lim_{\xi\to b} V(\xi)=0$ and $\lim_{\xi\to b} W(\xi)=0$. Therefore

$$< \partial_\varepsilon\phi(\xi,0), V(\xi)\times W(\xi) >= \sum_{k=-\infty}^{0}\det(P(\xi_k), V(\xi_k), W(\xi_k)) \; ,$$

where $\xi_k = F_0^k(\xi)$.

We perform a similar computation on the stable manifold, using a function $\psi : \mathcal{N}_0\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^3$ with the corresponding properties. For this function ψ, we conclude that

$$< \partial_\varepsilon\psi(\xi,0), V(\xi)\times W(\xi) >= -\sum_{k=1}^{\infty}\det(P(\xi_k), V(\xi_k), W(\xi_k)) \; .$$

Following a standard Melnikov argument [33], we conclude that if ξ^* is a nondegenerate zero of

$$M(\xi) =< \partial_\varepsilon\phi(\xi,0) - \partial_\varepsilon\psi(\xi,0), V(\xi)\times W(\xi) >$$

then near ξ^*, the two manifolds $W^u(F_\varepsilon)$ and $W^s(F_\varepsilon)$ intersect transversely.

It remains to show that each nondegenerate zero can be continued to a point in the primary intersection of the two manifolds. Let ξ^* be a nondegenerate zero of $M(\xi)$. Then, there is a curve $\zeta(\varepsilon)$ such that $\zeta(0)=\xi^*$ and, for all $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$,

$$\zeta(\epsilon)\in\phi(\mathcal{N}_0,\varepsilon)\cap\psi(\mathcal{N}_0,\varepsilon)\subset W^s(a,F_\varepsilon)\cap W^u(b,F_\varepsilon).$$

Now, we find fundamental domains $\mathcal{S},\mathcal{U}$ such that

$$\psi(\mathcal{N}_0,\varepsilon)\subset\mathcal{S}\subset W^s(a,F_\varepsilon)$$

and

$$\phi(\mathcal{N}_0,\varepsilon)\subset\mathcal{U}\subset W^s(a,F_\varepsilon)$$

and $\kappa(\mathcal{S},\mathcal{U})=0$ (cf. Defn. 1).

This implies that $\zeta(\varepsilon)$ is in the primary intersection of $W^s(a,F_\varepsilon)$ and $W^u(b,F_\varepsilon)$, and in this way, it can be continued with ε to the point ξ^*. Using a similar argument, it is possible to continue points in the primary intersection that are close to $\zeta(\varepsilon)$. ■

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