

# First-Order System Least Squares for Elastohydrodynamics with Application to Flow in Compliant Blood Vessels.

Jeffrey J. Heys, Curt G. DeGroat, Wendy W. Orlando,  
Thomas A. Manteuffel, Stephen F. McCormick

(JJH/TAM/SFM): Department of Applied Mathematics,  
University of Colorado at Boulder, Boulder, CO 80309

(CGD): UCHSC: The Children's Hospital  
Denver, CO 80218

(WWO): Northwest Research Associates Inc.,  
Colorado Research Associates Division, Boulder, CO 80301

## Abstract

Mathematical modeling of compliant blood vessels generally involves the Navier-Stokes equations on the evolving fluid domain and constitutive structural equations on the tissue domain. Coupling these systems while accounting for the changing shape of the fluid domain is a major challenge in numerical simulation. Many techniques have been developed to model compliant vessels, but all suffer from disproportionate increase in computational cost as problem complexity increases (i.e., larger domains, more dimensions, and more variables.). Even the best standard methods result in computational cost that typically grows quadratically with the degrees of freedom. Using a least-squares formulation of the problem, elliptic grid generation for the changing fluid domain, and an algebraic multigrid solver for the linear system can overcome many shortcomings of standard techniques. Most notably, the computational cost of solving the problem increases linearly with the degrees of freedom and the associated functional provides an *a posteriori* error measurement. Least squares represents a systematic approach for formulating the original problem so that the numerical process becomes straightforward and optimal, and it avoids restrictions that often limit other methods. We show results for a two-dimensional test problem consisting of a Newtonian fluid with properties of blood in a linear-elastic vessel with properties of smooth muscle. **Keywords:** Navier-Stokes, elasticity, coupled, finite elements, least squares, multigrid.

## Introduction

The mechanical coupling between a tissue and a moving bio-fluid is significant in many pathological conditions. For example, two conditions that we are interested in modeling are the blood flow through a total cavo-pulmonary connection (TCPC) and aqueous humor flow in the anterior eye. A brief description of these two systems follows.

A Right Heart Bypass (RHB) procedure (Fontan procedure) is performed in individuals with essentially no pulmonary ventricular pumping chamber. The RHB procedure reroutes the deoxygenated blood that is returning to the heart via the vena cavae directly into the pulmonary arteries,

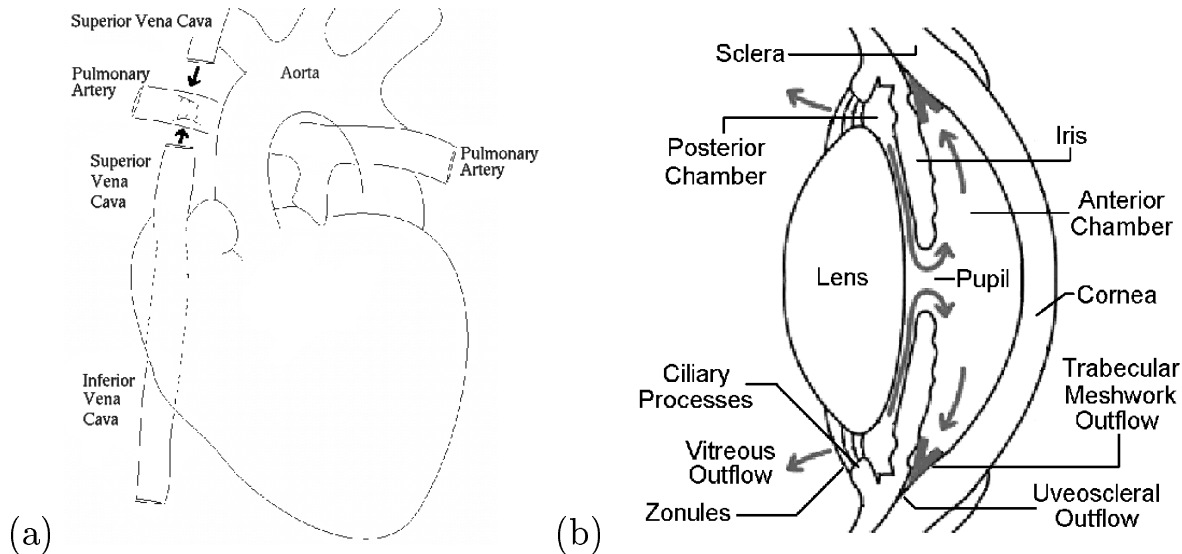


Figure 1: (b) A diagram of the anterior eye showing the path of aqueous humor circulation.

bypassing the heart. While there are different versions of the RHB procedure, the TCPC consists of a direct connection of the superior vena cava to the pulmonary arteries along with a conduit to connect the inferior vena cava to the pulmonary arteries (Figure 1a). Clinical evidence has shown that patients with a TCPC are sensitive to changes in their hemodynamic status [1]. Hence, extensive research has been directed at finding the most energy efficient geometric configuration for the TCPC to minimize stress on the lungs, heart, and blood vessels.

Nutrients and oxygen are delivered to the avascular tissues within the eye by a clear circulating fluid called Aqueous Humor (AH). AH is excreted by the ciliary bodies (Figure 1b), and it flows through the posterior chamber between the iris and lens. The AH then flows between the cornea and iris before exiting the anterior chamber through the vitreous meshwork. The flow of AH around the iris determines the passive deformation of the iris [2], and this passive deformation is critical in some forms of glaucoma [3].

Mathematical modeling of the TCPC and anterior eye is important for improving the treatment and/or understanding of these pathological conditions. A number of different techniques have been developed to accurately model the interaction between the moving bio-fluid and tissue. However, many of these techniques use iterative method to match the stresses between the tissue and bio-fluid [4], which allow errors to propagate, and the computation costs for even the best methods scale quadratically with problem size [5]. We present a First-Order System Least Squares (FOSLS) approach with at least two advantages over previous techniques: (1) the FOSLS function provides a *posteriori* error measure and (2) the system variables are essentially decoupled resulting in computation costs that increase nearly linearly with the degrees of freedom.

## Methods

Many tissues can be modeled as compressible elastic materials [6] using the following equation:

$$-\mu\Delta\mathbf{u} - (\lambda + \mu)\nabla\nabla \cdot \mathbf{u} = 0 \quad (1)$$

where  $\mu$  and  $\lambda$  are Lamé constants and  $\mathbf{u} = (u_1, u_2)^t$  is the displacement. Other material laws, such

as viscoelasticity, incompressible elasticity or biphasic, could be substituted with minimal effect on the numerical performance of the algorithm. AH, blood in large vessels, and many other bio-fluids can be modeled as incompressible Newtonian fluids using the Navier-Stokes equations:

$$-Re(\mathbf{v} \cdot \nabla \mathbf{v}) - \nabla p + \Delta \mathbf{v} = 0 \quad (2)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (3)$$

where  $Re$  is the Reynold's number,  $p$  is pressure, and  $\mathbf{v} = (v_1, v_2)^t$  is the velocity. The traction matching condition between the regions follows:

$$\mathbf{n} \cdot \sigma_{ij}(\mathbf{u}) = \mathbf{n} \cdot \sigma_{ij}(\mathbf{v}) \quad (4)$$

where  $\sigma_{ij}$  is the total stress tensor. The shape of the fluid domain is not known *a priori*, therefore we use Elliptic Grid Generation (EGG) to map the deformed fluid region back to the original rectangular region. The method requires the components of the inverse transformation to satisfy Laplace's equation:

$$\Delta \boldsymbol{\xi} = 0 \quad (5)$$

where  $\boldsymbol{\xi} = (\xi(x, y), \eta(x, y))^t$ .

Equations (1-5) can be recast as a first-order system of equations by defining new variables equal to the gradient of the primary variables. Rewriting the linear elasticity equation as a first-order system gives:

$$U - \nabla \mathbf{u} = 0 \quad (6)$$

$$-\mu \nabla \cdot U - (\lambda + \mu) \nabla tr(U) = 0 \quad (7)$$

$$\nabla \times U = 0 \quad (8)$$

Equation 8 is added to expose divergence free errors. The first-order system for the EGG mapping is

$$J - \nabla \mathbf{x} = 0 \quad (9)$$

$$(J^{-t} J^{-1} \nabla) \cdot J = 0 \quad (10)$$

$$\nabla \times J = 0 \quad (11)$$

where  $\mathbf{x} = (x, y)$  are the coordinates in the physical (deformed) space. Finally, the first-order system for the mapped Navier-Stokes equations is

$$V - \nabla \mathbf{v} = 0 \quad (12)$$

$$(J^{-t} J^{-1} \nabla) \cdot V - J^{-1} \nabla p_s - v \cdot J^{-1} V = 0 \quad (13)$$

$$\alpha \cdot tr(V) = 0 \quad (14)$$

$$\nabla \times V = 0 \quad (15)$$

where  $\alpha \rightarrow \infty$  by strictly enforcing the  $tr(U) = 0$  equation. The stress matching condition between the two regions can now be rewritten as:

$$J^{-1} \mathbf{n} \cdot (\mu(U + U^t) + \lambda \cdot tr(U) \cdot \mathbf{I}) - J^{-1} \mathbf{n} \cdot (V J^{-1} + (V J^{-1})^t - p_s \cdot \mathbf{I}) = 0 \quad (16)$$

The final functional is formed by summing the least squares norms of the residuals of equations ( 6 - 16), including the desired boundary terms. To find the minimum of the this *single* functional, the

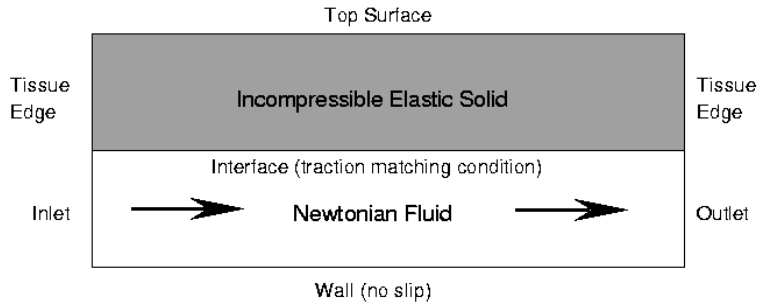


Figure 2: The 5 by 2 domain of the test problem, which consists of an elastic solid above a Newtonian fluid. The top of the solid is fixed.

first Fréchet derivative is set to zero, and then Newton’s method is applied to the weak form [7]. The weak form is solved using the finite element method (bilinear basis functions), and the resulting linear system is solved using an Algebraic Multigrid (AMG) method [8].

### Test Problem

The 2-dimensional domain of the test problem is shown in Figure 2. A no slip condition is imposed on the bottom and the top of the fluid domain, and the dimensionless normal stress is set to 2 on the inlet and 0 at the outlet. The  $Re$  of the fluid is set to 300, a typical value for blood flow [9]. The upper surface of the elastic region was fixed. On the sides of the elastic region, the displacement was set to zero in the x-direction and the stress was set to zero in the y-direction. Unless otherwise specified,  $\lambda$  is set to 9.3 and  $\mu$  is equal to 1.0, corresponding to a Poisson’s ratio of 0.45 and a dimensionless Young’s modulus of 3.0.

## Results

The flow profile and elastic deformation resulting from solving the fully-coupled, steady-state test problem are shown in Figure 3. The highest pressure (i.e., highest normal stress) are at the inlet of the flow region causing the elastic solid to be displaced upward. As the flow continues, the pressure

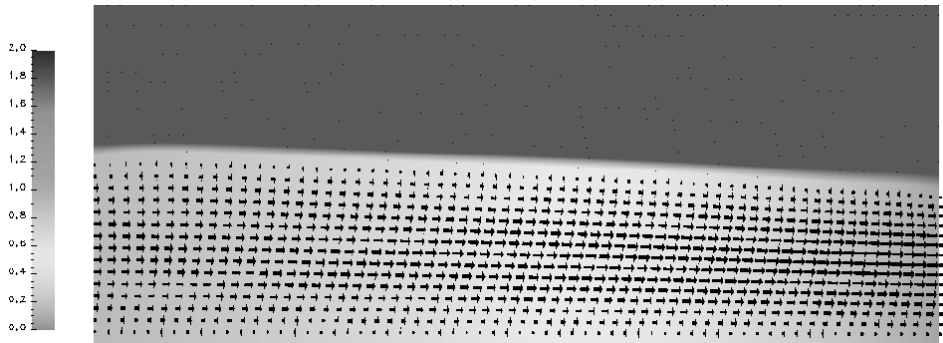


Figure 3: The flow profile and displacement of the test problem using a 64x32 element grid

Table 1: Computational performance using a FOSLS formulation on the test problem with different meshes

Mesh	Convergence Factor	Functional	CPU Time
32x16	0.82	0.035	90 sec
64x32	0.84	0.011	327 sec
128x64	0.85	0.0039	1166 sec

drops and the fluid displacement is decreased.

Table 1 describes the computation performance associated with using the FOSLS functional and solving the linear system with an AMG solver. The convergence factor is defined as the fractional reduction in the function for each V-cycle iteration. One of the advantages of the FOSLS formulation is that the size of the functional provides a sharp error measurement. As shown in Table 1, the size of the functional decreases as the mesh is refined. Further, the size of the functional on each element can be used as an error measurement for adaptive refinement. The Degrees of Freedom (DOF) in the problem can be calculated based on the mesh dimensions and using the fact that there are 12 DOF per node in the fluid domain and 8 DOF per node in the elastic domain. Using the CPU time required to minimize the functional, the relationship between the total DOF and CPU time can be described by the equation,

$$\text{CPU time} = 0.024 \cdot \text{DOF}^{0.96} \quad (17)$$

which shows that by using the FOSLS formulation, the computation costs scale linearly with the problem size.

By mapping the physical fluid domain back to a rectangular domain using EGG, large deformations do not increase the computation costs significantly, unlike methods that iteratively match the stress. Figure 4 shows the mesh deformation when the Young's modulus of the elastic region is reduced by a factor of 3 and the Poisson's ratio is reduced to 0.40. The CPU time required was the same as the standard test problem.

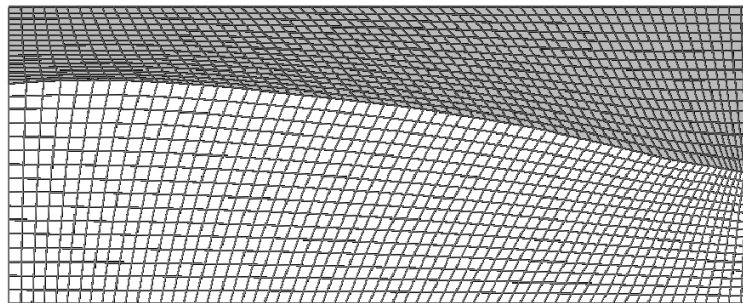


Figure 4: The distorted mesh resulting from solving a coupled fluid-elastic problem with a dimensionless Young's modulus of 1 and a Poisson's ration of 0.4

## Conclusions

A FOSLS finite element formulation of the original PDE's in conjunction with an AMG solver results in a straightforward and optimal numerical process. First, the FOSLS functional provides an accurate measure of the error in the discrete solution. Second, the computation cost of solving the problem increases linearly (*not* quadratically) with increasing problem size. The results presented in this paper are intended to demonstrate the effectiveness of the FOSLS formulation on a simple test problem that is not unlike blood flow in a vessel. We plan to extend this work to 3 spatial dimensions, and model pulsatile flow in the TCPC region.

## References

- [1] M. Rosenthal, A. Bush, J. Deanfield, and A. Redington, "Comparison of cardiopulmonary adaptation during exercise in children after the atriopulmonary and total cavo-pulmonary Fontan procedures," *Circulation*, vol. 91, pp. 372–8, 1995.
- [2] Jeffrey J. Heys, Victor H. Barocas, and Michael J. Taravella, "Modeling passive mechanical interaction between aqueous humor and iris," *Journal of Biomechanical Engineering*, vol. 123, no. 6, pp. 540–547, 2001.
- [3] John R. Karickhoff, "Pigmentary dispersion syndrome and pigmentary glaucoma: A new mechanism concept, a new treatment, and a new technique," *Ophthalmic Surgery*, vol. 23, no. 4, pp. 269–277, 1992.
- [4] Vinod B. Makhijani, H. Q. Yang, Paul J. Dionne, and Mano J. Thubrikar, "Three-dimensional coupled fluid-structure simulation of pericardial bioprosthetic aortic valve function," *ASAIO Journal*, vol. 43, pp. M387–M392, 1997.
- [5] Jeffrey J. Heys, *An Elastohydrodynamic Model of the Anterior Eye with Application to Pigmentary Glaucoma*, PhD thesis, University of Colorado at Boulder, 2001.
- [6] Jeffrey Heys and Victor H. Barocas, "Mechanical characterization of the bovine iris," *Journal of Biomechanics*, vol. 32, no. 9, pp. 999–1003, 1999.
- [7] A. L. Codd, *Elasticity-Fluid Coupled Systems and Elliptic Grid Generation (EGG) based on First-Order system Least Squares (FOSLS)*, PhD thesis, University of Colorado at Boulder, 2001.
- [8] J. W. Ruge and K. Stuben, "Algebraic multigrid," in *Multigrid Methods*, Stephen F. McCormick, Ed., vol. 3 of *Frontiers in Applied Mathematics*, pp. 73–130. SIAM, Philadelphia, 1987.
- [9] K. Perktold, E. Thurner, and Th. Kenner, "Flow and stress characteristics in rigid walled and compliant carotid artery bifurcation models," *Medical and Biological Engineering and Computing*, vol. 32, pp. 19–26, 1994.