FIRST-ORDER SYSTEM LEAST SQUARES AND ELECTRICAL IMPEDANCE TOMOGRAPHY: PART I

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Abstract. The concept of Electrical Impedance Tomography (EIT) is to create an image by estimating an interior electromagnetic parameter such that it is consistent with a set of boundary data. The standard approach to EIT is output least squares (OLS). For a set of applied normal boundary currents, one minimizes the defect between the measured and computed boundary voltages associated, respectively, with the exact impedance and its approximation. In minimizing a boundary functional, OLS implicitly imposes the governing Poisson equation as an optimization constraint. We introduce a new first-order system least-squares (FOSLS) formulation that incorporates the elliptic PDE as an interior functional in a global minimization scheme. We establish equivalence of our functional to OLS and to an existing interior least-squares functional due to Kohn and Vogelius. That the latter may be viewed as a special dual approach (FOSLL*) suggests FOSLS as a unifying framework for EIT. This sets the stage for broad applicaton of the theory presented in our companion paper [27]

1. Introduction.

1.1. Overview. Electrical Impedance Tomography (EIT) belongs to a family of imaging methods based on distinguishing interior spatial variation of an electromagnetic parameter. The basic principle is to approximate this parameter so that it is consistent with given pairings of boundary current and boundary voltage data. In mathematical parlance, this boundary data represents discrete knowledge of the Dirichlet-to-Neumann map associated with the unknown parameter. The broad scope of EIT's application, ranging from biomedical imaging [9, 12, 3] to geophysical prospecting [34, 28, 38, 15] to industrial process tomography [31, 16, 18], implies the importance of qualitatively different parameter regimes. Thus, it is reasonable to expect efficient and effective algorithms to be highly problem specific. The central motivation for our study, both here and in [27], is the rigorous analysis of what can be rightfully reconstructed and its incumbent incorporation into an efficient numerical scheme that admits the use of problem specific information. By directly appealing to the physics, we are able to better understand the sense in which this notoriously ill-posed problem can be posed well. This is in keeping with the convictions put forth by Backus and Gilbert in their now classical theory on geophysical inverse problems [1]. We feel that revisiting these convictions, in light of recent advances in least-squares techniques, leads us to important and fundamentally new contributions.

In this paper, our perspective is initiated with discussion of the governing equations, the standard least-squares approach, and established theoretical insights. The focus is then on proving equivalence of three least-squares EIT formulations: Output Least Squares [9], a formulation due to Kohn and Vogelius [25], and a new FOSLS formulation. That the Kohn and Vogelius functional may be viewed as a first-order system LL^* (FOSLL^* [7]) formulation suggests FOSLS as a unifying framework for EIT. In our companion study [27], we identify and accept EIT's inherent limitations on resolution with an analysis of the sense in which error in conductivity is reduced when minimizing any functional within the FOSLS framework. Also in the compan-

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ion paper, preliminary numerical results for the FOSLS formulation, which is derived below, are shown to be faithful to the theory presented here.

1.2. Mathematical setting. The development of a mathematical model for the EIT inverse problem begins with Maxwell's equations in a source-free, closed, and simply connected domain, which we denote Ω with boundary Γ. A series of assumptions (see [9], [30], and references therein), including negligible magnetic permeability and quasistatic periodic time variation, leads to a Poisson equation for complex-valued voltage p:

$$\nabla \cdot \gamma \nabla p = 0 \,, \tag{1.1}$$

where $\gamma = \sigma + i\omega\epsilon$ is the admittance. EIT refers to the reconstruction of this parameter, since electrical impedance is defined as $1/\gamma$.

The theory we establish first makes the electrostatic assumption, $\omega = 0$. Thus, the quantities in (1.1) are real valued, and the aim is reconstruction of the conductivity, σ, based on its associated Dirichlet and Neumann boundary data. Equivalently, one could reconstruct the resistivity, $1/\sigma$, the case of electrostatic current. The electrostatic assumption is an important first principle since the lack of resolution with quasistatic fields, and thus EIT, is attributed to the conductivity term. Extension to electromagnetic fields varying slowly in time is left for future work.

REMARK 1. *We refer to the process of reconstructing conductivity given Dirichlet and Neumann data as electrical conductivity tomography (ECT), and remark that in the literature it is also referred to as the DC resistivity problem, applied potential tomography, electric current computed tomography, and electrical resistance tomography.*

In what follows, we adopt the usual Sobolev spaces, $\mathrm{L}^2(\Omega)$ and $\mathrm{H}^1(\Omega)$, including the dual boundary spaces, $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$ [13]. We assume that the boundary, Γ, is Lipschitz and note that the normal current flux is the scaled Neumann data, $\mathbf{n} \cdot \sigma\nabla p$, where $\mathbf{n}$ denotes the outward unit normal. Also, the boundary voltage distribution is the Dirichlet data, $p_{|_\Gamma}$. It is most practical to impose the normal current and measure the resulting voltage data, and we choose this perspective in the presentation of our theory. Hence, considering the Neumann problem

$$\begin{aligned} \nabla \cdot \sigma \nabla p &= 0 \,, \\ \mathbf{n} \cdot \sigma \nabla p &= h \,, \end{aligned} \tag{1.2}$$

where the normal current $h \in \mathrm{H}^{-1/2}(\Gamma)$ satisfies $\int_\Gamma h \, ds = 0$, one may view coefficient σ as determining a Neumann-to-Dirichlet map; i.e., as determining the trace of voltage $p_{|_\Gamma} \in \mathrm{H}^{1/2}(\Gamma)$ on the boundary (up to a constant). An alternative and more realistic mathematical model, the complete model [9, 41], uses Robin boundary conditions in place of the Dirichlet conditions as the measured data. The theory we develop does not preclude, along with other important practical matters, the choice of the type of imposed boundary data.

Several definitions associated with system (1.2) now follow to facilitate the formal statement of the inverse problem of ECT.

DEFINITION 1.1. *Denote* $\mathrm{L}^\infty_+(\Omega) := \{\sigma \in \mathrm{L}^\infty(\Omega) : \sigma > 0\}$. *Also, let*

$$\mathrm{H}^{1/2}_0(\Gamma) := \left\{ g \in \mathrm{H}^{1/2}(\Gamma) : \int_\Gamma g \, d\tau = 0 \right\} ,$$

$$\mathrm{H}^{-1/2}_0(\Gamma) := \left\{ h \in \mathrm{H}^{-1/2}(\Gamma) : \int_\Gamma h \, d\tau = 0 \right\} .$$

A given conductivity, $\sigma \in \mathrm{L}^\infty_+(\Omega)$, induces a bounded linear map,

$$\mathcal{R}_\sigma : \mathrm{H}_0^{-1/2}(\Gamma) \longrightarrow \mathrm{H}_0^{1/2}(\Gamma) ,$$

from the imposed normal current to the resulting boundary voltage. This map is defined by

$$\mathcal{R}_\sigma(h) := p_{|\Gamma} ,$$

where p solves (1.2), and we call $\mathcal{R}_\sigma$ the Neumann-to-Dirichlet map (NtD map) associated with σ.

REMARK 2. *Function space $\mathrm{L}^\infty_+(\Omega)$ is used for admissible conductivity since it is the least restrictrive for which the NtD map associated with (1.2) can be defined. However, uniquely defining the NtD requires further restriction, to be discussed in the next section. Regarding the boundary spaces, a compatibility condition on (1.2) implies that we must have $\int_\Gamma h\, d\tau = 0$. Similarly, p is determined by (1.2) only up to a constant, so we are permitted to impose the condition $\int_\Gamma g\, d\tau = 0$.*

DEFINITION 1.2. *We say that $\{p, \mathcal{R}_\sigma(h), h\} \in \mathrm{H}^1(\Omega) \times \mathrm{H}_0^{1/2}(\Gamma) \times \mathrm{H}_0^{-1/2}(\Gamma)$ is a Dirichlet-Neumann triple (DN triple) for σ if (1.2) holds. Also, we shall refer to $\{\mathcal{R}_\sigma(h), h\}$ as a DN pair.*

REMARK 3. *The NtD map, $\mathcal{R}_\sigma$, is self-adjoint in the L^2-inner-product. This is apparent since for DN triples $\{p_1, \mathcal{R}_\sigma(h_1), h_1\}$ and $\{p_2, \mathcal{R}_\sigma(h_2), h_2\}$, integration by parts leads to*

$$\langle h_1, \mathcal{R}_\sigma(h_2)\rangle_{0,\Gamma} = \int_\Omega \sigma \nabla p_1 \cdot \nabla p_2 \, dx \; = \langle \mathcal{R}_\sigma(h_1), h_2\rangle_{0,\Gamma}.$$

Finally, we have the notation to define the inverse problem of ECT.

DEFINITION 1.3. *The inverse problem of electrical conductivity tomography (ECT) is to approximate the exact conductivity, σ^*, according to a given graph of measured Dirichlet-Neumann data $\{g_i, h_i\}_{i=1}^L$, where $g_i \approx \mathcal{R}_{\sigma^*}(h_i)$.*

Stated otherwise, the inverse problem of ECT is to approximately invert the nonlinear mapping $\sigma \longrightarrow \mathcal{R}_\sigma$. The approximate nature of this inversion is due to the pre-imposed discreteness of finite data, the inexactness of this data, and the need to choose a discrete approximation space for conductivity. Next, we give background on these issues as they affect our ability to solve the inverse problem. Ultimately, however, we defer an examination of these issues to the companion paper [27].

1.3. Theoretical background. Classical discussion of existence and uniqueness of the inverse problem is hindered by several concerns. First, assuming exact boundary data is the same as assuming the existence of a conductivity consistent with the boundary data. However, given the intrinsic error in the measurements of the data, it is likely that there is no such consistent conductivity. As for uniqueness, progressively refined results have been established [4, 5, 24, 20, 35, 36]. When the conductivity, σ, is assumed to be continuous, uniqueness for 2D is proved in [5] only under the assumption that $\nabla\sigma \in \mathrm{L}^p$, where $p > 2$. Separately derived in [20], uniqueness for a discontinuous conductivity is established assuming only that σ, piecewise, is in C^2. However, the techniques used in these theoretical findings rely on complete and precise boundary information and cannot accomodate the reality of only a finite number of approximate DN pairs. In our companion paper [27], the theory we present quantifies how inexact and incomplete boundary data unleash error in σ that cannot be confidently eliminated. Moreover, the ultimate statement of our theory is valid for $\sigma \in L^\infty_+(\Omega)$, the most general space of admissible conductivity.

To begin to address the difficulty of posing the inverse problem of ECT, it is instructive to consider Calderon's initial analysis [8]. Let $\delta \in L^1(\Omega)$ be a perturbation of a background conductivity, σ. For DN triples $\{p_i, \mathcal{R}_\sigma(h_i), h_i\}$ and $\{p_j, \mathcal{R}_\sigma(h_j), h_j\}$, he observes that

$$\langle h_i, \mathcal{R}_{\sigma+\delta}(h_j) - \mathcal{R}_\sigma(h_j)\rangle_{0,\Gamma} = \int_\Omega \delta\, \nabla p_i \cdot \nabla p_j \, dx \; + \; O(\delta^2)\,. \tag{1.3}$$

This classical expansion displays how perturbations in the interior can be practically undetectable at the boundary. For example, observe that, with $\sigma = 1$, the distribution of voltage, p, is increasingly smooth away from the boundary. Thus, the product of the gradients of solutions to the Laplace equation places decreasing weight on perturbations of conductivity according to distance into the interior. This weighting is central to the coercive bound we establish in [27] that guides an understanding of which components of conductivity can be reconstructed. However, we avoid an explicit linearization to achieve this.

In now classical work, Backus and Gilbert [1] introduce the notion of "inference" for studying a wide class of geophysical inverse problems, not just ECT. Parker [29] has applied this theory to ECT, considering the linearized one-dimensional problem of determining the conductive layers of stratified earth. Independently, Seagar [32, 33] assesses the "visibility" of circular discs of constant conductivity as perturbations of a background constant conductivity in a 2D circular domain by using conformal mapping techniques. To determine whether the effect of a certain error can be detected on the boundary, Isaacson [19] introduce "distinguishability." Rather than survey and bridge these approaches, we begin from first principles and make frequent allusion to the language and concepts of each throughout our analysis. Ultimately, the result is a more general and flexible analysis.

In line with all of these resolution analyses, the best test to apply in order to signal a specific error, $\sigma - \sigma^*$, is that which attains the operator norm of the defect in the NtD maps; i.e., that which attains the sup in

$$\|\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*}\|_{\mathcal{L}(H_0^{-1/2}(\Gamma), H_0^{1/2}(\Gamma))} := \sup_{h \in H_0^{-1/2}(\Gamma)} \frac{\|(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})h\|_{1/2,\Gamma}}{\|h\|_{-1/2,\Gamma}}\,. \tag{1.4}$$

Given complete and exact boundary data, the natural functional to minimize is the operator norm in (1.4). Of course, with finite data, the sup can only be taken over a finite-dimensional subspace, $span\{h_i\}_{i=1}^L \subset H_0^{-1/2}(\Gamma)$, where L is the number of available DN data. Moreover, this data is only approximate. Despite both the pre-imposed discreteness and error in the data, we may still recover meaningful components of σ^*. To reduce the number of forward problems solved to determine $\mathcal{R}_\sigma(h)$, the applied boundary tests may be adaptively chosen so as to approximate the test that achieves the above operator norm [10, 14, 17].

1.4. Output Least Squares. In analogy to the Frobenius norm of matrix algebra, a natural alternative to minimizing the sup over the span of the boundary data is to minimize the Output Least-Squares (OLS) functional,

$$\mathcal{F}(\sigma; \{g_i, h_i\}_{i=1}^L) := \sum_{i=1}^L \|\mathcal{R}_\sigma(h_i) - g_i\|^2_{1/2,\Gamma}. \tag{1.5}$$

This formulation is an example of unconstrained least-squares optimization and is the standard approach throughout the wide range of EIT applications [2, 9, 15, 41]. We remark that the L^2-norm, rather than $H^{1/2}$-norm, is typically enforced. Also, the Dirichlet data is usually interpolated as constant on the support of the electrode and unknown elsewhere. However, throughout our study, we preserve the most general Sobolev space setting.

To minimize $\mathcal{F}$ in (1.5), the first variation of the NtD map with respect to σ enables a standard Newton quadraticization. The first-order term in (1.3) is used to represent this variation, determining the effect on the boundary of changing a given degree of freedom for σ. The associated linear system to be solved at each Newton step is known as the "sensitivity matrix". Once a single Newton step is performed to obtain a new approximation, the L forward problems are solved and the process is repeated (see [9] and references therein). The physics at the source of EIT's difficulty is manifested by the extremely high condition number of the sensitivity matrix when a uniform grid for σ is chosen. This is consistent with what the physics suggests: that an increasingly coarse representation of conductivity should be used as the grid extends into the interior. Typically, some form of regularization is instead performed to ensure that the Hessian is sufficiently positive definite.

Regularization refers to the process of posing an inverse problem well [37], and has taken several forms in application to EIT. A regularization operator, $\mathcal{B}$, can be incorporated into the OLS functional as an additional least-squares term, so that (1.5) is replaced by

$$\mathcal{F}(\sigma; \{g_i, h_i\}_{i=1}^{L}) := \left(\sum_{i=1}^{L} \|\mathcal{R}_\sigma(h_i) - g_i\|_{0,\Gamma}^2 \right) + \alpha \|\mathcal{B}\sigma\|_{0,\Omega}^2 , \tag{1.6}$$

where α is a regularization parameter that is usually chosen empirically to balance resolution and computability in some sense. The most common regularization is a penalty on the H^1-seminorm of σ [9], wherein $\mathcal{B} = \nabla$. When piecewise constant basis functions are used in the discrete representation of σ, one can impose a penalty on the jumps between each element [11]. Naturally, the resulting reconstruction is artificially smoothed, especially near the boundary where one ought to be able to recover high frequency components. However, the technique has been actively researched for many years, and ways of reducing the artificial smoothness have been developed, including the recent multilevel approach of Borcea [2]. The theory we develop suggests that the FOSLS formulation inherently imposes a smoothing of the conductivity estimate that is relative to the distance from the boundary in accordance with the governing physics.The FOSLS formulation may likewise benefit from an empirical process of determining ideal weightings on each term in the functional, analogous to determining α in (1.6). Hence, this presents a potential direction for future research.

An alternative regularization scheme, referred to as a statistical or Bayesian approach, is implemented by Vauhkonen et al. [40, 21, 22] to penalize the orthogonal complement of an operator which represents prior information. The penalty thus serves to steer the approximation toward an expected solution. A related method, that of basis constraints [39], builds the subspace of admissible conductivity, which may in practice be of very low dimension, by using basis functions selected according to *a priori* structural information. Though we endorse the use of basis constraints to incorporate prior information into the solution process, it is our conviction that such information should not serve to "regularize" the inverse problem. Rather, one must accept and account for the lack of resolution that the governing physics mandates.

The theory developed in [27] is flexible enough to analyze the extent to which error can be eliminated when employing any reasonable space to approximate the unknown conductivity.

In the next section, we prove equivalence of the Output Least Squares formulation to two interior least-squares functionals. Ultimately, we show how these methods fit into what may be viewed as a FOSLS framework. This sets the stage for wide application of the theory we develop in our companion paper [27].

2. FOSLS Framework.

In what follows, we write $\mathcal{R}_{\sigma^*}(h)$ instead of g to stress the assumption on zero error in the data, as well as the dependence on σ^*. Analysis of error in the data is deferred to [27]. Finally, we develop the framework explicitly for 2D and anticipate that the setting generalizes to 3D, although we draw attention to those points in the development for which the 3D generalization is not obvious.

2.1. Kohn and Vogelius.

The first interior least-squares formulation we address was introduced by Kohn and Vogelius [25, 23] and is closely related to the original interior functional proposed for ECT by Wexler [42]. We now review their formulation in 2D before proving its equivalence to OLS.

2.1.1. Formulation.

To begin, we recall the interior partial differential equation (PDE) and boundary conditions associated with the exact, albeit unknown, conductivity, σ^*:

$$\begin{aligned} \nabla \cdot \sigma^* \nabla p &= 0 && \text{in } \Omega\,, \\ \mathbf{n} \cdot \sigma^* \nabla p &= h && \text{on } \Gamma\,, \\ p &= \mathcal{R}_{\sigma^*}(h) && \text{on } \Gamma\,. \end{aligned}$$

Letting $\nabla^\perp = (-\partial_y, \partial_x)^t$, and recalling that Ω is simply connected, then the divergence-free condition on $\sigma^* \nabla p$ asserted in this PDE implies the existence of an $s \in \mathrm{H}^1(\Omega)$, depending on σ^*, h, and Ω, such that

$$\sigma^* \nabla p = \nabla^\perp s\,.$$

On Γ, we have

$$\mathbf{n} \cdot \sigma^* \nabla p = \mathbf{n} \cdot \nabla^\perp s = \boldsymbol{\tau} \cdot \nabla s = h\,,$$

where $\boldsymbol{\tau}$ denotes the clockwise unit tangent vector. Note that s satisfies

$$\begin{aligned} \nabla \cdot \frac{1}{\sigma^*} \nabla s = \nabla \times \frac{1}{\sigma^*} \nabla^\perp s &= 0 && \text{in } \Omega\,, \\ s &= \int_\Gamma h\, d\gamma && \text{on } \Gamma\,. \end{aligned}$$

To assess the validity of an approximate conductivity, the Kohn and Vogelius (KV) functional is based on defining two separate sets of Dirichlet problems: given $\{\mathcal{R}_{\sigma^*}(h_i), h_i\}_{i=1}^L$, find $p_i \in \mathrm{H}^1(\Omega)$ such that

$$\begin{aligned} \nabla \cdot \sigma \nabla p_i &= 0 && \text{in } \Omega\,, \\ p_i &= \mathcal{R}_{\sigma^*}(h_i) && \text{on } \Gamma\,, \end{aligned} \tag{2.1}$$

and $s_i \in \mathrm{H}^1(\Omega)$ such that

$$\begin{aligned} \nabla \times \frac{1}{\sigma} \nabla^{\perp} s_i &= 0 && \text{in } \Omega\,, \\ s_i &= \int_{\Gamma} h_i \, d\gamma && \text{on } \Gamma\,. \end{aligned} \tag{2.2}$$

Now, if $\sigma = \sigma^*$, then

$$\|\sigma \nabla p_i - \nabla^{\perp} s_i\|_{0,\Omega} = 0 \qquad \forall\, i \in \{1, 2, \ldots, L\}\,. \tag{2.3}$$

With a complete set of boundary data associated with σ^*, the converse statement, which confirms uniqueness, is also true: if (2.1) - (2.3) are satisfied for every DN pair $\{\mathcal{R}_{\sigma^*}(h), h\}$, then $\sigma = \sigma^*$. To our knowledge, this converse statement has not been proved in the literature, but it follows directly from the equivalence result we establish together with uniqueness theory.

The statement in (2.3) suggests minimizing the functional

$$\mathcal{F}_{KV}(\sigma; \{\mathcal{R}_{\sigma^*}(h_i), h_i\}_{i=1}^{L}) := \sum_{i=1}^{L} \left(\min_{p_i, s_i \in \mathrm{H}^1(\Omega)} \|\sigma^{1/2} \nabla p_i - \sigma^{-1/2} \nabla^{\perp} s_i\|_{0,\Omega}^2 \right)\,, \tag{2.4}$$

where, for each $i = 1, 2, \ldots, L$, we define

$$\begin{aligned} p_{i|_{\Gamma}} &:= \mathcal{R}_{\sigma^*}(h_i)\,, \\ s_{i|_{\Gamma}} &:= \int_{\Gamma} h_i \, d\gamma\,. \end{aligned} \tag{2.5}$$

The rescaling of conductivity used in (2.4) is chosen so that, for a fixed approximation, σ, the minimizations in (2.4) can be conducted separately. This fact is shown explicitly in Lemma 2.2 below.

The scaling of conductivity used in (2.4) also benefits the updating process for conductivity. For a fixed set of approximations, $\{p_i, s_i\}_{i=1}^{L}$, it is shown in [23] that the best conductivity is determined pointwise to be

$$\sigma = \left(\sum_{i=1}^{L} |\, \nabla^{\perp} s_i \,|^2 \right)^{1/2} \left(\sum_{i=1}^{L} |\, \nabla p_i \,|^2 \right)^{-1/2}\,.$$

This sets up a natural minimization process of alternating between the subspace for σ and those for $\{p_i, s_i\}_{i=1}^{L}$. Indeed, this alternating process is the design of the proof of the equivalence result. In numerical studies of the KV functional in [23], minimization in σ is incorporated explicitly and, instead of the natural alternating scheme, a Newton quadraticization is used.

2.1.2. Equivalence to Output Least Squares. Here we establish equivalence of the KV functional to that of Output Least Squares (OLS). This enables us to focus on OLS in [27]. Because the line of proof follows for each DN pair, it is convenient to denote

$$\mathrm{F}(\sigma, p, s) := \|\sigma^{1/2} \nabla p - \sigma^{-1/2} \nabla^{\perp} s\|_{0,\Omega}\,.$$

Also, we observe that, for any DN pair $\{\mathcal{R}_{\sigma^*}(h), h\}$, the exact conductivity generates solutions p^* and s^* to (2.1) and (2.2), respectively, such that $\mathrm{F}(\sigma^*, p^*, s^*) = 0$. For any arbitrary $p\,, s \in \mathrm{H}^1(\Omega)$ satisfying (2.5), we can interpret $\mathrm{F}(\sigma, p, s)$ as an expression of error in each variable. Thus, letting $\mathrm{H}_0^1(\Omega) := \{q \in \mathrm{H}^1(\Omega) : q = 0 \text{ on } \Gamma\}$, it is useful to define

$$\begin{aligned} \delta p &:= p - p^* \ \in \ \mathrm{H}_0^1(\Omega)\ , \\ \delta s &:= s - s^* \ \in \ \mathrm{H}_0^1(\Omega)\ . \end{aligned}$$

Note that since δp and δs are zero on the boundary, we preserve the boundary values of p^* and s^*, as required by the functional, and we can rewrite (2.4) subject to (2.5) as

$$\mathcal{F}_{KV}(\sigma; \{\mathcal{R}_{\sigma^*}(h_i), h_i\}_{i=1}^L) = \sum_{i=1}^L \left(\min_{\delta p_i, \delta s_i \in \mathrm{H}_0^1(\Omega)} \mathrm{F}(\sigma, p_i^* + \delta p_i, s_i^* + \delta s_i) \right) . \tag{2.6}$$

The equivalence relation we seek, Corollary 2.3 below, follows from a generalized ellipticity property for functional term $\mathrm{F}(\sigma, p^* + \delta p, s^* + \delta s)$. Essentially, in the following theorem we establish that $\mathrm{F}(\sigma, p^* + \delta p, s^* + \delta s)$ is equivalent to an expression of specific quantities $\mathcal{R}_\sigma(h) - \mathcal{R}_{\sigma^*}(h)$, $\delta p - \delta p_D$, and $\delta s - \delta s_D$, where

$$\begin{aligned} \delta p_D &:= \arg \min_{\delta p \,\in\, \mathrm{H}_0^1(\Omega)} \|\sigma^{1/2} \nabla (p^* + \delta p)\|_{0,\Omega}\ , \\ \delta s_D &:= \arg \min_{\delta s \,\in\, \mathrm{H}_0^1(\Omega)} \|\sigma^{-1/2} \nabla^\perp (s^* + \delta s)\|_{0,\Omega} \end{aligned} \tag{2.7}$$

come from the minimizations in (2.6). Hence by design, conducting the minimizations in (2.6) leads to lower and upper bounds for $\mathcal{F}_{KV}(\sigma; \{\mathcal{R}_{\sigma^*}(h_i), h_i\}_{i=1}^L)$ in terms of the sampling of defects in the NtD map.

THEOREM 2.1. *Suppose that $\sigma \in \mathrm{L}_+^\infty(\Omega)$ satisfies $\sigma = \sigma^*$ on Γ. Suppose also that $s^* \in \mathrm{H}^1(\Omega)$ is such that $\mathrm{F}(\sigma^*, p^*, s^*) = 0$ for DN triple $\{p^*, \mathcal{R}_{\sigma^*}(h), h\}$ associated with σ^*, where $h \in \mathrm{H}_0^{-1/2}(\Gamma)$. Finally, let δp_D and δs_D be as defined in (2.7). Then there exist positive constants c_0 and c_1 such that*

$$\begin{aligned} c_0 \|(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})h\|_{1/2,\Gamma}^2 + \|\sigma^{1/2}\nabla(\delta p - \delta p_D)\|_{0,\Omega}^2 + \|\sigma^{-1/2}\nabla^\perp(\delta s - \delta s_D)\|_{0,\Omega}^2 \le \\ F(\sigma, p^* + \delta p, s^* + \delta s) \\ \le c_1 \|(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})h\|_{1/2,\Gamma}^2 + \|\sigma^{1/2}\nabla(\delta p - \delta p_D)\|_{0,\Omega}^2 + \|\sigma^{-1/2}\nabla^\perp(\delta s - \delta s_D)\|_{0,\Omega}^2\ , \end{aligned} \tag{2.8}$$

where c_0 depends on Ω and $\min_\Omega |\sigma|$, $c_1 = \|\mathcal{R}_\sigma^{-1}\|_{\mathcal{L}(\mathrm{H}_0^{1/2}(\Gamma), \mathrm{H}_0^{-1/2}(\Gamma))}$, and each only multiply the first terms.

Proof. The optimality of δp_D and δs_D, characterized in (2.7), leads to the orthogonality conditions

$$\begin{aligned} \langle \sigma \nabla (p^* + \delta p_D), \nabla \delta p \rangle_{0,\Omega} = 0 \qquad & \forall\, \delta p \,\in\, \mathrm{H}_0^1(\Omega)\ , \\ \langle \sigma^{-1} \nabla^\perp (s^* + \delta s_D), \nabla^\perp \delta s \rangle_{0,\Omega} = 0 \qquad & \forall\, \delta s \,\in\, \mathrm{H}_0^1(\Omega)\ . \end{aligned} \tag{2.9}$$

Note that the Neumann data for δp_D and δs_D are generally not zero and that, in preserving the Dirichlet data, the strong form of the first equation in (2.9) means that $\{p^* + \delta p_D, \mathcal{R}_\sigma(h), h\}$ is a DN triple. Now, we expand the terms of the functional, use (2.9), and note that all but one cross term vanishes upon integrating by parts:

$$
\begin{aligned}
&F(\sigma, p^* + \delta p, s^* + \delta s) \\
&\qquad = \|\sigma^{1/2}\nabla(p^* + \delta p_D + \delta p - \delta p_D) - \sigma^{-1/2}\nabla^\perp(s^* + \delta s_D + \delta s - \delta s_D)\|_{0,\Omega}^2 \\
&\qquad = \|\sigma^{1/2}(\nabla p^* + \delta p_D)\|_{0,\Omega}^2 + \|\sigma^{1/2}\nabla(\delta p - \delta p_D)\|_{0,\Omega}^2 - 2\langle \nabla p^*, \nabla^\perp s^*\rangle_{0,\Omega} \\
&\qquad\quad + \|\sigma^{-1/2}\nabla^\perp(s^* + \delta s_D)\|_{0,\Omega}^2 + \|\sigma^{-1/2}\nabla^\perp(\delta s - \delta s_D)\|_{0,\Omega}^2 \, . \qquad (2.10)
\end{aligned}
$$

By assumption, $\sigma^*\nabla p^* = \nabla^\perp s^*$, so we may split the lone cross term in (2.10) to write

$$
\begin{aligned}
F(\sigma, p^* + \delta p, s^* + \delta s) &= \|\sigma^{1/2}\nabla(p^* + \delta p_D)\|_{0,\Omega}^2 - \|{\sigma^*}^{1/2}\nabla p^*\|_{0,\Omega}^2 \\
&\quad + \|\sigma^{-1/2}\nabla^\perp(s^* + \delta s_D)\|_{0,\Omega}^2 - \|{\sigma^*}^{-1/2}\nabla^\perp s^*\|_{0,\Omega}^2 \\
&\quad + \|\sigma^{1/2}\nabla(\delta p - \delta p_D)\|_{0,\Omega}^2 + \|\sigma^{-1/2}\nabla^\perp(\delta s - \delta s_D)\|_{0,\Omega}^2 \, . \qquad (2.11)
\end{aligned}
$$

With the last two terms as desired, consider the other four. The second equation in (2.9) implies that $\sigma^{-1}\nabla^\perp(s^* + \delta s_D)$ is equal to a grad. Thus, there exists a $\delta p_N \in \mathrm{H}^1(\Omega)$ such that

$$
\sigma\nabla(p^* + \delta p_N) = \nabla^\perp(s^* + \delta s_D) \, .
$$

Together with the Dirichlet data for $s = s^* + \delta s_D$ in (2.5), we have

$$
\mathbf{n}\cdot\sigma\nabla(p^* + \delta p_N) = \mathbf{n}\cdot\nabla^\perp(s^* + \delta s_D) = h \, . \qquad (2.12)
$$

In other words, we may think of $\{p^* + \delta p_N, \mathcal{R}_\sigma(h), h\}$ as a DN triple for σ. This allows us to write (2.11) as

$$
\begin{aligned}
F(\sigma, p^* + \delta p, s^* + \delta s) &= \|\sigma^{1/2}\nabla(p^* + \delta p_D)\|_{0,\Omega}^2 - \|{\sigma^*}^{1/2}\nabla p^*\|_{0,\Omega}^2 \\
&\quad + \|\sigma^{1/2}\nabla(p^* + \delta p_N)\|_{0,\Omega}^2 - \|{\sigma^*}^{1/2}\nabla p^*\|_{0,\Omega}^2 \\
&\quad + \|\sigma^{1/2}\nabla(\delta p - \delta p_D)\|_{0,\Omega}^2 + \|\sigma^{-1/2}\nabla^\perp(\delta s - \delta s_D)\|_{0,\Omega}^2 \, . \qquad (2.13)
\end{aligned}
$$

Next, by integration parts yields the following relations:

$$
\begin{aligned}
\langle \mathcal{R}_{\sigma^*}(h), h\rangle_{0,\Gamma} &= \|{\sigma^*}^{1/2}\nabla p^*\|_{0,\Omega}^2 \, , \\
\langle \mathcal{R}_{\sigma}(h), h\rangle_{0,\Gamma} &= \|\sigma^{1/2}\nabla(p^* + \delta p_N)\|_{0,\Omega}^2 \, , \\
\left\langle \mathcal{R}_{\sigma^*}^{-1}(\mathcal{R}_{\sigma^*}(h)), \mathcal{R}_{\sigma^*}(h)\right\rangle_{0,\Gamma} &= \|{\sigma^*}^{1/2}\nabla p^*\|_{0,\Omega}^2 \, , \\
\left\langle \mathcal{R}_{\sigma}^{-1}(\mathcal{R}_{\sigma^*}(h)), \mathcal{R}_{\sigma^*}(h)\right\rangle_{0,\Gamma} &= \|\sigma^{1/2}\nabla(p^* + \delta p_D)\|_{0,\Omega}^2 \, . \qquad (2.14)
\end{aligned}
$$

Thus, (2.13) becomes

$$
\begin{aligned}
F(\sigma, p^* + \delta p, s^* + \delta s) &= \left\langle (\mathcal{R}_\sigma^{-1} - \mathcal{R}_{\sigma^*}^{-1})\mathcal{R}_{\sigma^*}(h), \mathcal{R}_{\sigma^*}(h)\right\rangle_{0,\Gamma} + \langle (\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h), h\rangle_{0,\Gamma} \\
&\quad + \|\sigma^{1/2}\nabla(\delta p - \delta p_D)\|_{0,\Omega}^2 + \|\sigma^{-1/2}\nabla^\perp(\delta s - \delta s_D)\|_{0,\Omega}^2 \, . \qquad (2.15)
\end{aligned}
$$

Since $\mathcal{R}_\sigma$ and $\mathcal{R}_{\sigma^*}$ are self-adjoint, we may combine the first two terms:

$$
\begin{aligned}
&\left\langle (\mathcal{R}_\sigma^{-1} - \mathcal{R}_{\sigma^*}^{-1})\mathcal{R}_{\sigma^*}(h), \mathcal{R}_{\sigma^*}(h)\right\rangle_{0,\Gamma} + \langle (\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h), h\rangle_{0,\Gamma} \\
&\qquad = \left\langle (\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*} + \mathcal{R}_{\sigma^*}\mathcal{R}_\sigma^{-1}\mathcal{R}_{\sigma^*} - \mathcal{R}_{\sigma^*})(h), h\right\rangle_{0,\Gamma} \\
&\qquad = \left\langle (\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})\mathcal{R}_\sigma^{-1}(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h), h\right\rangle_{0,\Gamma} \\
&\qquad = \left\langle \mathcal{R}_\sigma^{-1}(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h), (\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h)\right\rangle_{0,\Gamma} \, . \qquad (2.16)
\end{aligned}
$$

Also, since

$$\begin{aligned}\langle \mathcal{R}_\sigma^{-1}(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h), (\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})h\rangle_{0,\Gamma} \\ \le \|\mathcal{R}_\sigma^{-1}\|_{\mathcal{L}(\mathrm{H}_0^{1/2}(\Gamma),\mathrm{H}_0^{-1/2}(\Gamma))}\|(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h)\|^2_{1/2,\Gamma},\end{aligned}$$

we may combine (2.15) and (2.16) to arrive at the desired upper bound:

$$\begin{aligned}\mathrm{F}(\sigma, p^* + \delta p, s^* + \delta s) \le \|\mathcal{R}_\sigma^{-1}\|_{\mathcal{L}(\mathrm{H}_0^{1/2}(\Gamma),\mathrm{H}_0^{-1/2}(\Gamma))} \|\mathcal{R}_\sigma(h) - \mathcal{R}_{\sigma^*}(h)\|^2_{1/2,\Gamma} \\ + \|\sigma^{1/2}\nabla(\delta p - \delta p_D)\|^2_{0,\Omega} + \|\sigma^{-1/2}\nabla^\perp(\delta s - \delta s_D)\|^2_{0,\Omega}\,.\end{aligned}$$

To achieve the lower bound, we recall (2.7) and observe that

$$(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h) = \delta p_{N|_\Gamma} \;\in\; \mathrm{H}_0^{-1/2}(\Gamma)$$

by definition of δp_N. Hence, we may now consider the $\tilde{p} \in \mathrm{H}^1(\Omega)$ that solves

$$\begin{aligned}\nabla\cdot\sigma\nabla\tilde{p} &= 0 && \text{in } \Omega\,,\\ \tilde{p} &= \delta p_N && \text{on } \Gamma\,.\end{aligned}$$

Appealing to (2.14) and the Trace Theorem [13], we have

$$\begin{aligned}\langle \mathcal{R}_\sigma^{-1}(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h), (\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h)\rangle_{0,\Gamma} &= \|\sigma^{1/2}\nabla\tilde{p}\|^2_{0,\Omega}\\ &\ge c\,\|\nabla\tilde{p}\|^2_{0,\Omega}\\ &\ge c_0\,\|\delta p_N\|^2_{1/2,\Gamma}\\ &= c_0\|(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h)\|^2_{1/2,\Gamma}\,, \qquad (2.17)\end{aligned}$$

where c_0 depends on the smoothness of the boundary and $\min_\Omega |\sigma|$. Finally, incorporating (2.17) with (2.16) into (2.6) establishes the lower bound:

$$\begin{aligned}\mathrm{F}(\sigma, p^* + \delta p, s^* + \delta s) \ge c_0\|(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h)\|^2_{1/2,\Gamma} \\ + \|\sigma^{1/2}\nabla(\delta p - \delta p_D)\|^2_{0,\Omega} + \|\sigma^{-1/2}\nabla^\perp(\delta s - \delta s_D)\|^2_{0,\Omega}\,.\end{aligned}$$

□

To make use of the above equivalence, we establish the following lemma using the constructs of the theorem to reassert the ability to minimize the functional in (2.4) for s_i and p_i separately.

LEMMA 2.2. *Suppose $\sigma = \sigma^*$ on Γ. Also, for each $h \in \mathrm{H}_0^{-1/2}(\Gamma)$, let $\{p^*, \mathcal{R}_{\sigma^*}(h), h\}$ be a DN triple for σ^* and define*

$$\begin{aligned}\delta p_D &:= \arg\min_{\delta p\,\in\,\mathrm{H}_0^1(\Omega)} \|\sigma^{1/2}\nabla(p^* + \delta p)\|_{0,\Omega}\,,\\ \delta s_D &:= \arg\min_{\delta s\,\in\,\mathrm{H}_0^1(\Omega)} \|\sigma^{-1/2}\nabla^\perp(s^* + \delta s)\|_{0,\Omega}\,. \qquad (2.18)\end{aligned}$$

Then

$$\mathrm{F}(\sigma, p^* + \delta p_D, s^* + \delta s_D) = \min_{\delta p, \delta s\,\in\,\mathrm{H}_0^1(\Omega)} \mathrm{F}(\sigma, p^* + \delta p, s^* + \delta s)\,. \qquad (2.19)$$

Proof. From (2.10) and the boundary data for p^* and s^*, we have

$$\begin{aligned}
&F(\sigma, p^* + \delta p, s^* + \delta s) \\
&\quad = \|\sigma^{1/2}(\nabla p^* + \delta p_D)\|_{0,\Omega}^2 + \|\sigma^{1/2}\nabla(\delta p - \delta p_D)\|_{0,\Omega}^2 - 2\langle \nabla p^*, \nabla^\perp s^* \rangle_{0,\Omega} \\
&\qquad + \|\sigma^{-1/2}\nabla^\perp(s^* + \delta s_D)\|_{0,\Omega}^2 + \|\sigma^{-1/2}\nabla^\perp(\delta s - \delta s_D)\|_{0,\Omega}^2 \\
&\quad = \|\sigma^{1/2}(\nabla p^* + \delta p_D)\|_{0,\Omega}^2 + \|\sigma^{1/2}\nabla(\delta p - \delta p_D)\|_{0,\Omega}^2 - 2\langle \mathcal{R}_{\sigma^*}(h), h \rangle_{0,\Gamma} \\
&\qquad + \|\sigma^{-1/2}\nabla^\perp(s^* + \delta s_D)\|_{0,\Omega}^2 + \|\sigma^{-1/2}\nabla^\perp(\delta s - \delta s_D)\|_{0,\Omega}^2 .
\end{aligned}$$

The conclusion is apparent since the boundary term is a known fixed quantity. ∎

A direct result of this lemma and the product equivalence established in Theorem 2.1 is the following corollary.

COROLLARY 2.3. *For any $h \in \mathrm{H}_0^{-1/2}(\Gamma)$, there exist positive constants c_0 and c_1 such that*

$$\begin{aligned}
&c_0\|\mathcal{R}_\sigma(h) - \mathcal{R}_{\sigma^*}(h)\|_{1/2,\Gamma}^2 \\
&\qquad \le \min_{p,s\,\in\,\mathrm{H}^1(\Omega)} \|\sigma^{1/2}\nabla p - \sigma^{-1/2}\nabla^\perp s\|_{0,\Omega}^2 \\
&\qquad\qquad \le c_1\|\mathcal{R}_\sigma(h) - \mathcal{R}_{\sigma^*}(h)\|_{1/2,\Gamma}^2 , \qquad (2.20)
\end{aligned}$$

where c_0 depends on Ω and $\min_\Omega|\sigma|$, $c_1 = \|\mathcal{R}_\sigma^{-1}\|_{\mathcal{L}(\mathrm{H}_0^{1/2}(\Gamma),\mathrm{H}_0^{-1/2}(\Gamma))}$, and where p and s are subject to the boundary data in (2.5) for h,

It is also clear from Lemma 2.2, as a generalization of the above corollary, that the Kohn and Vogelius formulation is equivalent to the Output Least Squares formulation. We state this formally in the following corollary.

COROLLARY 2.4. *With $\mathcal{F}_{KV}$ defined in (2.4) subject to (2.5), there exist positive constants c_0 and c_1 such that*

$$\begin{aligned}
&c_0\sum_{i=1}^{L}\|\mathcal{R}_\sigma(h_i) - \mathcal{R}_{\sigma^*}(h_i)\|_{1/2,\Gamma}^2 \\
&\qquad \le \mathcal{F}_{KV}(\sigma; \{\mathcal{R}_{\sigma^*}(h_i), h_i\}_{i=1}^L) \\
&\qquad\qquad \le c_1\sum_{i=1}^{L}\|\mathcal{R}_\sigma(h_i) - \mathcal{R}_{\sigma^*}(h_i)\|_{1/2,\Gamma}^2 , \qquad (2.21)
\end{aligned}$$

where $c_1 = \|\mathcal{R}_\sigma^{-1}\|_{\mathcal{L}(\mathrm{H}_0^{1/2}(\Gamma),\mathrm{H}_0^{-1/2}(\Gamma))}$ and c_0 depends on Ω and $\min_\Omega|\sigma|$.

This corollary allows us to focus on the Output Least Squares formulation in [27], ensuring that the Kohn and Vogelius formulation inherits the sense in which the inverse problem can be posed well.

REMARK 4. *As a final remark, consider (2.21) under the assumption of complete knowledge of the Dirichlet map. It is then natural to take the sup over all $h \in \mathrm{H}_0^{-1/2}(\Gamma)$ of each quantity in (2.20). Scaling appropriately then leads to the idealized equivalence result*

$$\begin{aligned}
&c_0\|\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*}\|_{\mathcal{L}(\mathrm{H}_0^{-1/2}(\Gamma),\mathrm{H}_0^{1/2}(\Gamma))}^2 \\
&\qquad \le \sup_{h\in\mathrm{H}_0^{-1/2}(\Gamma)} \frac{\left(\min_{p,s\,\in\,\mathrm{H}^1(\Omega)}\|\sigma^{1/2}\nabla p - \sigma^{-1/2}\nabla^\perp s\|_{0,\Omega}^2\right)}{\|h\|_{-1/2,\Gamma}} \\
&\qquad\qquad \le c_1\|\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*}\|_{\mathcal{L}(\mathrm{H}_0^{-1/2}(\Gamma),\mathrm{H}_0^{1/2}(\Gamma))}^2 ,
\end{aligned}$$

where, again, p and s are subject to the boundary data in (2.5) for a given h. Hence, we may imagine a sup version of the Kohn and Vogelius functional whose minimization, our theory shows, is equivalent to minimizing the operator norm of the defect in the ND maps.

2.2. FOSLS. We now present a first-order system least-squares (FOSLS) formulation for ECT. The first-order system we use may be viewed as a rescaled version of the Maxwell equations. While it is evident that this formulation retains our ability to solve the forward problem fast, the FOSLS formulation also provides a remarkably insightful process for updating the approximate conductivity, again setting up a natural alternating minimization scheme. We now present the FOSLS formulation in 3D and show that the 2D version is also equivalent to Output Least Squares.

2.2.1. Formulation. To derive the FOSLS formulation for ECT, define $q := \frac{1}{2}\log(\sigma)$, so that $e^{2q} := \sigma$, and let $\mathbf{u} := e^q \nabla p$. Then, with $\{p, \mathcal{R}_\sigma(h), h\}$ a DN triple for σ, consider the first-order system

$$\begin{aligned} e^{-q}\nabla \cdot e^q \mathbf{u} &= 0 && \text{in } \Omega\,, \\ e^{q}\nabla \times e^{-q} \mathbf{u} &= 0 && \text{in } \Omega\,, \\ \mathbf{n} \cdot e^q \mathbf{u} &= h && \text{on } \Gamma\,. \end{aligned} \tag{2.22}$$

We may think of (2.22) as determining the tangential current

$$\mathbf{n} \times e^{-q}\mathbf{u} = (\mathbf{n} \times \nabla)\mathcal{R}_\sigma(h) \, \in \, (\mathrm{H}_0^{-1/2}(\Gamma))^{n-1}.$$

When $n = 2$, we write $(\boldsymbol{\tau}\cdot)$ in lieu of $(\mathbf{n}\times)$. We now make our first definition regarding the FOSLS formulation.

DEFINITION 2.5. *First-order system (2.22) defines the normal-to-tangential current map (NtT map), $\mathcal{P}_q : \mathrm{H}_0^{-1/2}(\Gamma) \longrightarrow (\mathrm{H}_0^{-1/2}(\Gamma))^{n-1}$, where*

$$\mathcal{P}_q(h) = \mathbf{n} \times e^{-q}\mathbf{u} = \, (\mathbf{n} \times \nabla)\mathcal{R}_\sigma(h)\,.$$

We also define the following variations on $\mathrm{H}(div, \Omega)$ and $\mathrm{H}(curl, \Omega)$.

DEFINITION 2.6. *For $q \in \mathrm{L}^\infty(\Omega)$, let $\mathrm{W}_q = \mathrm{H}(div\ e^q, \Omega) \cap \mathrm{H}(curl\ e^{-q}, \Omega)$, where*

$$\begin{aligned} \mathrm{H}(div\ e^q, \Omega) &= \{\mathbf{u} \, \in \, (\mathrm{L}^2(\Omega))^n \, : \, \nabla \cdot e^q \mathbf{u} \, \in \, \mathrm{L}^2(\Omega)\}\,, \\ \mathrm{H}(curl\ e^{-q}, \Omega) &= \left\{\mathbf{u} \, \in \, (\mathrm{L}^2(\Omega))^n \, : \, \nabla \times e^{-q}\mathbf{u} \, \in \, (\mathrm{L}^2(\Omega))^{2n-3}\right\}, \end{aligned}$$

for $n = 2, 3$.

Finally, it is convenient to introduce an analogy to the DN-triple notation.

DEFINITION 2.7. *In the FOSLS setting, if $\mathbf{u}$ solves (2.22) with boundary data h, then we refer to $\{\mathbf{u}, h, \mathcal{P}_{q^*}(h)\}$ as a normal-tangential triple (NT triple) for q and to $\{h, \mathcal{P}_{q^*}(h)\}$ as an NT pair.*

Given the boundary data, $\{h_i, \mathcal{P}_{q^*}(h_i)\}_{i=1}^L$, associated with an unknown q^*, we approximate q^* by minimizing the FOSLS functional

$$\mathcal{F}_{OSLS}(q; \{h_i, \mathcal{P}_{q^*}(h_i)\}_{i=1}^L) = \sum_{i=1}^{L} \left(\min_{\mathbf{u}_i \,\in\, \mathrm{W}_q} \mathrm{F}(q, \mathbf{u}_i; h_i, \mathcal{P}_{q^*}(h_i)) \right)\,, \tag{2.23}$$

where

$$\begin{aligned} \mathrm{F}(q, \mathbf{u}_i; h_i, \mathcal{P}_{q^*}(h_i)) := {}& \|e^{-q}\nabla \cdot e^q \mathbf{u}_i\|_{0,\Omega}^2 + \|e^q \nabla \times e^{-q}\mathbf{u}_i\|_{0,\Omega}^2 \\ &+ \|\mathbf{n} \cdot e^q \mathbf{u}_i - h_i\|_{-1/2,\Gamma}^2 + \|\mathbf{n} \times e^{-q}\mathbf{u}_i - \mathcal{P}_{q^*}(h_i)\|_{-1/2,\Gamma}^2\,. \end{aligned} \tag{2.24}$$

To minimize $\mathcal{F}_{OSLS}$, first note that, for a fixed approximation, q, each $\mathbf{u}_i$ can be solved with optimal efficiency [6]. Furthermore, if we assume $q \in \mathrm{H}^1(\Omega)$ and $\mathbf{u} \in \mathrm{W}_0 = \mathrm{H}(div, \Omega) \cap \mathrm{H}(curl, \Omega)$, then for fixed $\mathbf{u}_i \in (\mathrm{H}^1(\Omega))^n$, we also have an elliptic problem to improve approximation $q \in \mathrm{H}^1(\Omega)$. Taking the first variation in q of $\mathcal{F}_{OSLS}$ for a fixed set, $\{\mathbf{u}_i \in (\mathrm{H}^1(\Omega))^2\}$, reveals that the best $q \in \mathrm{H}^1(\Omega)$ satisfies the weak form

$$\langle A\nabla q, \nabla r\rangle_{0,\Omega} = f(r) \qquad \forall\, r \in \mathrm{H}^1(\Omega), \tag{2.25}$$

where

$$A = 2\sum_{i=1}^{L} |\mathbf{u}_i|^2 \tag{2.26}$$

and $f \in \mathrm{H}^{-1}(\Omega)$ is a known function dependent on each $\mathbf{u}_i$. For example, if $\mathbf{u}_i \in \mathrm{H}^2(\Omega)$, then $f(r) = \langle f, r\rangle_{0,\Omega}$, where

$$f = \sum_{i=1}^{L} \left[(\nabla \cdot \mathbf{u}_i)^2 + (\nabla \times \mathbf{u}_i)^2 + \mathbf{u}_i^t \boldsymbol{\Delta} \mathbf{u}_i \right].$$

These properties suggest that we can employ an alternating subspace minimization scheme. Furthermore, it is remarkable that the minimization associated with the weak form in (2.25) incorporates a weighting on the components of q that proves natural to ECT. Because requiring $q \in \mathrm{H}^1(\Omega)$ is quite restrictive, the OLS or Kohn and Vogelius approach may be better suited for many applications. However, the role of FOSLS as a framework for EIT is compelling, and its exploration is warranted out of theoretical concern alone.

2.2.2. Equivalence to Output Least Squares. As with the Kohn and Vogelius functional, we establish equivalence for $n = 2$. To show that the FOSLS functional in (2.23) is equivalent to OLS, we first establish several lemmas and some useful notation. In the first lemma, we relate $\mathcal{P}_q$ to $\mathcal{R}_\sigma$ in 2D. Generalization of this lemma to the 3D problem is not obvious, and not addressed in this study. Complication arises since $\mathcal{P}_q : \mathrm{R}^1 \to \mathrm{R}^2$, whereas $\mathcal{R}_\sigma(h)$ is a scalar quantity.

LEMMA 2.8. *Let $\sigma = \sigma^*$ on Γ and $n = 2$. Then*

$$\|\mathcal{P}_q - \mathcal{P}_{q^*}\|_{\mathcal{L}(\mathrm{H}_0^{-1/2}(\Gamma), \mathrm{H}_0^{-1/2}(\Gamma))} = \|\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*}\|_{\mathcal{L}(\mathrm{H}_0^{-1/2}(\Gamma), \mathrm{H}_0^{1/2}(\Gamma))}. \tag{2.27}$$

Proof. We wish to show that

$$\sup_{h \in \mathrm{H}_0^{-1/2}(\Gamma)} \frac{\|(\mathcal{P}_q - \mathcal{P}_{q^*})(h)\|_{-1/2,\Gamma}}{\|h\|_{-1/2,\Gamma}} = \sup_{h \in \mathrm{H}_0^{-1/2}(\Gamma)} \frac{\|(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h)\|_{1/2,\Gamma}}{\|h\|_{-1/2,\Gamma}}.$$

Since $(\mathcal{P}_q - \mathcal{P}_{q^*})(h) = (\boldsymbol{\tau} \cdot \nabla)(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h)$, it suffices to show that

$$\|(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h)\|_{1/2,\Gamma} = \|(\boldsymbol{\tau} \cdot \nabla)(\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})(h)\|_{-1/2,\Gamma}.$$

Noting that there is a $g \in \mathrm{H}^{1/2}(\Gamma)$ such that $g = (\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*})h$, then this reduces to

$$\|g\|_{1/2,\Gamma} = \|\frac{dg}{d\tau}\|_{-1/2,\Gamma}. \tag{2.28}$$

We now show that this holds for any $g \in H_0^{1/2}(\Gamma)$ by appealing to the representation of the $H^{1/2}$- and $H^{-1/2}$-norms using the Fourier Transform. Defining $\hat{g}(\omega)$ as the transform of g, we have [26]

$$\begin{aligned} \|g\|_{1/2,\Gamma} &= \|\omega^{1/2}\hat{g}\|_{0,\Gamma} \,, \\ \|h\|_{-1/2,\Gamma} &= \|\omega^{-1/2}\hat{h}\|_{0,\Gamma} \,. \end{aligned}$$

Hence, (2.28) follows from

$$\|g\|_{1/2,\Gamma} = \|\omega^{1/2}\hat{g}\|_{0,\Gamma} = \|\omega^{-1/2}\omega\hat{g}\|_{0,\Gamma} = \|\omega^{-1/2}\hat{g'}\|_{0,\Gamma} = \|\frac{dg}{d\tau}\|_{-1/2,\Gamma} \,.$$

□

It is convenient to isolate the interior and boundary functional terms of the FOSLS functional in (2.23) and write

$$\begin{aligned} \mathrm{F}(q,\mathbf{u}_i;h,\mathcal{P}_{q^*}(h_i)) &:= \mathrm{F_I}(q,\mathbf{u}_i) + \mathrm{F_B}(q,\mathbf{u}_i;h_i,\mathcal{P}_{q^*}(h_i)) \,, \\ \mathrm{F_I}(q,\mathbf{u}_i) &:= \|e^{-q}\nabla\cdot e^{q}\mathbf{u}_i\|^2_{0,\Omega} + \|e^{q}\nabla\times e^{-q}\mathbf{u}_i\|^2_{0,\Omega} \,, \\ \mathrm{F_B}(q,\mathbf{u}_i;h_i,\mathcal{P}_{q^*}(h_i)) &:= \|\mathbf{n}\cdot e^{q}\mathbf{u}_i - h_i\|^2_{-1/2,\Gamma} + \|\boldsymbol{\tau}\cdot e^{-q}\mathbf{u}_i - \mathcal{P}_{q^*}(h_i)\|^2_{-1/2,\Gamma} \,. \end{aligned} \tag{2.29}$$

The following lemmas set the stage for stating and proving the equivalence results. The first establishes a useful equivalence relation between F and the norm defined on $\mathrm{W}_q(\Omega)$ by

$$\|\mathbf{w}\|^2_{\mathrm{W}_q} := \|\mathbf{w}\|^2_{0,\Omega} + \|e^{-q}\nabla\cdot e^{q}\mathbf{w}\|^2_{0,\Omega} + \|e^{q}\nabla\times e^{-q}\mathbf{w}\|^2_{0,\Omega} \,.$$

.

LEMMA 2.9. *Let $\mathbf{w} \in \mathrm{W}_q(\Omega)$. Then there exists a constant, c, depending on q and Ω, such that*

$$c_0\|\mathbf{w}\|^2_{\mathrm{W}_q} \le \mathrm{F}(q,\mathbf{w};0,0) \le c_1\|\mathbf{w}\|^2_{\mathrm{W}_q} \,. \tag{2.30}$$

Proof. The upper bound in (2.30) follows directly from the Trace Theorem [13]. For the lower bound, it suffices to show there is a c such that

$$\begin{aligned} &\|\mathbf{w}\|^2_{0,\Omega} \le \\ &\quad c\left(\|e^{-q}\nabla\cdot e^{q}\mathbf{w}\|^2_{0,\Omega} + \|e^{q}\nabla\times e^{-q}\mathbf{w}\|^2_{0,\Omega} + \|\mathbf{n}\cdot e^{q}\mathbf{w}\|^2_{-1/2,\Gamma} + \|\boldsymbol{\tau}\cdot e^{-q}\mathbf{w}\|^2_{-1/2,\Gamma}\right) \,. \end{aligned}$$

Then, c_0 can be chosen as $1/c$. First, we use a scaled Helmholtz decomposition [13]: $\forall\, \mathbf{v} \in (\mathrm{L}^2(\Omega))^2$, there exist $\phi \in \mathrm{H}^1_0(\Omega)$ and $\psi \in \mathrm{H}^1(\Omega)/\mathbb{R}$ such that

$$\mathbf{v} = e^{q}\nabla\phi + e^{-q}\nabla^{\perp}\psi \,.$$

The logic leading to this representation is to choose ϕ so that $\nabla\cdot e^{2q}\nabla\phi = \nabla\cdot e^{q}\mathbf{v}$, and to similarly choose ψ to satisfy $\nabla\times e^{-2q}\nabla^{\perp}\psi = \nabla\times e^{-q}\mathbf{v}$. The choice of boundary conditions on ϕ and ψ enables us to eliminate the cross term that would otherwise appear below, and to use a Poincaré-Friedrichs inequality. With this decomposition, we write

$$\begin{aligned}
\|\mathbf{w}\|_{0,\Omega} &= \sup_{\mathbf{v}\,\in\,(\mathrm{L}^2(\Omega))^2} \frac{\langle \mathbf{w}, \mathbf{v}\rangle_{0,\Omega}}{\|\mathbf{v}\|_{0,\Omega}} \\
&= \sup_{\phi,\psi\,\in\,\mathrm{H}^1_0(\Omega)} \frac{\langle \mathbf{w}, e^q\nabla\phi + e^{-q}\nabla^\perp\psi\rangle_{0,\Omega}}{\|e^q\nabla\phi + e^{-q}\nabla^\perp\psi\|_{0,\Omega}} \\
&\le \sup_{\phi\,\in\,\mathrm{H}^1_0(\Omega)} \frac{\langle \mathbf{w}, e^q\nabla\phi\rangle_{0,\Omega}}{\|e^q\nabla\phi\|_{0,\Omega}} + \sup_{\psi\,\in\,\mathrm{H}^1_0(\Omega)} \frac{\langle \mathbf{w}, e^{-q}\nabla^\perp\psi\rangle_{0,\Omega}}{\|e^{-q}\nabla^\perp\psi\|_{0,\Omega}}\,.
\end{aligned} \tag{2.31}$$

Applying Green's theorem to the first term, we have

$$\begin{aligned}
&\frac{\langle \mathbf{w}, e^q\nabla\phi\rangle_{0,\Omega}}{\|e^q\nabla\phi\|_{0,\Omega}} \\
&\quad\le \frac{|\langle e^{-q}\nabla\cdot e^q\mathbf{w}, e^q\phi\rangle_{0,\Omega}| + |\langle \mathbf{n}\cdot e^q\mathbf{w}, \phi\rangle_{0,\Gamma}|}{\|e^q\nabla\phi\|_{0,\Omega}} \\
&\quad\le \frac{\|e^{-q}\nabla\cdot e^q\mathbf{w}\|_{0,\Omega}\|e^q\phi\|_{0,\Omega} + \|\mathbf{n}\cdot e^q\mathbf{w}\|_{-1/2,\Gamma}\|\phi\|_{1/2,\Gamma}}{\|e^q\nabla\phi\|_{0,\Omega}} \\
&\quad\le \left(\|e^{-q}\nabla\cdot e^q\mathbf{w}\|^2_{0,\Omega} + \|\mathbf{n}\cdot e^q\mathbf{w}\|^2_{-1/2,\Gamma}\right)^{1/2} \left(\frac{\|e^q\phi\|^2_{0,\Omega} + \|\phi\|^2_{1/2,\Gamma}}{\|e^q\nabla\phi\|^2_{0,\Omega}}\right)^{1/2}\,.
\end{aligned} \tag{2.32}$$

The term in (2.32) that does not involve $\mathbf{w}$ may be bounded by a constant depending on q and Ω. To see this, note that we can use the Poincaré-Friedrichs inequality [13] to write

$$\frac{\|e^q\phi\|^2_{0,\Omega} + \|\phi\|^2_{1/2,\Gamma}}{\|e^q\nabla\phi\|^2_{0,\Omega}} \le \frac{\max_\Omega\{e^{2q},1\}}{\min_\Omega e^{2q}} \frac{\left(\|\phi\|^2_{0,\Omega} + \|\phi\|^2_{1/2,\Gamma}\right)}{\|\nabla\phi\|^2_{0,\Omega}} \le c_2(q)$$

where

$$c_2(q) = c_2\frac{\max_\Omega\{e^{2q},1\}}{\min_\Omega e^{2q}}\,. \tag{2.33}$$

Thus, (2.32) becomes

$$\sup_{\phi\,\in\,\mathrm{H}^1_0(\Omega)} \frac{\langle \mathbf{w}, e^q\nabla\phi\rangle_{0,\Omega}}{\|e^q\nabla\phi\|_{0,\Omega}} \le c_2(q)\left(\|e^{-q}\nabla\cdot e^q\mathbf{w}\|^2_{0,\Omega} + \|\mathbf{n}\cdot e^q\mathbf{w}\|^2_{-1/2,\Gamma}\right)^{1/2}\,. \tag{2.34}$$

Similarly, there is a $c_3(q)$ such that

$$\sup_{\psi\,\in\,\mathrm{H}^1_0(\Omega)} \frac{|\,\langle \mathbf{w}, e^{-q}\nabla^\perp\psi\rangle_{0,\Omega}\,|}{\|e^{-q}\nabla^\perp\psi\|_{0,\Omega}} \le c_3(q)\left(\|e^q\nabla\times e^{-q}\mathbf{w}\|^2_{0,\Omega} + \|\boldsymbol{\tau}\cdot e^{-q}\mathbf{w}\|^2_{-1/2,\Gamma}\right)^{1/2}\,. \tag{2.35}$$

Therefore, combining (2.34) and (2.35) and choosing c appropriately proves the lemma. $\square$

We also make use of the next lemma to relate the FOSLS functional directly to the NtT map.

Lemma 2.10. *Let* $\mathbf{w} \in \mathrm{W}_q(\Omega)$ *and suppose* $\mathbf{n}\cdot e^q\mathbf{w} = h$ *on* Γ. *If* $\mathbf{w}_h$ *satisfies*

$$\begin{aligned}
e^{-q}\nabla\cdot e^q\mathbf{w}_h &= 0 && \text{in } \Omega\,, \\
e^q\nabla\times e^{-q}\mathbf{w}_h &= 0 && \text{on } \Gamma\,, \\
\mathbf{n}\cdot e^q\mathbf{w}_h &= h && \text{on } \Gamma\,,
\end{aligned} \tag{2.36}$$

then there exists a constant, c, depending only on Ω*, such that* $\boldsymbol{\tau}\cdot e^{-q}(\mathbf{w}-\mathbf{w}_h)$ *satisfies*

$$\|\boldsymbol{\tau}\cdot e^{-q}(\mathbf{w}-\mathbf{w}_h)\|^2_{-1/2,\Gamma} \le c\,\mathrm{F}_I(q,\mathbf{w}) \ . \tag{2.37}$$

Proof. We have from the Trace Theorem [13] that

$$\begin{aligned}\|\boldsymbol{\tau}\cdot e^{-q}(\mathbf{w}-\mathbf{w}_h)\|^2_{-1/2,\Gamma} &\le c\left(\|\nabla\times e^{-q}(\mathbf{w}-\mathbf{w}_h)\|^2_{0,\Omega} + \|e^{-q}(\mathbf{w}-\mathbf{w}_h)\|^2_{0,\Omega}\right)\\ &\le c\left(\|\nabla\times e^{-q}(\mathbf{w}-\mathbf{w}_h)\|^2_{0,\Omega} + \|e^{q}(\mathbf{w}-\mathbf{w}_h)\|^2_{0,\Omega}\right) \ ,\end{aligned}$$

where c depends on Ω and $\max_\Omega e^q$. Since, by assumption, $\mathbf{n}\cdot e^q((\mathbf{w}-\mathbf{w}_h))=0$ on the boundary, we may use a Poincaré-Friedrichs inequality to rewrite the L^2 term to find that

$$\begin{aligned}\|\boldsymbol{\tau}\cdot e^{-q}(\mathbf{w}-\mathbf{w}_h)\|^2_{-1/2,\Gamma} &\le c\left(\|\nabla\times e^{-q}(\mathbf{w}-\mathbf{w}_h)\|^2_{0,\Omega} + \|\nabla\cdot e^{q}(\mathbf{w}-\mathbf{w}_h)\|^2_{0,\Omega}\right)\\ &\le\ c\,\mathrm{F}_I(q,\mathbf{w})\,,\end{aligned}$$

where, again, c depends on Ω and $\max_\Omega e^q$. $\square$

The above lemmas now allow us to establish a lower bound for the FOSLS functional in terms of the defect in the ND map. In analogy to the equivalence established for the Kohn and Vogelius formulation, the line of proof follows for each NT pair. To begin, let $\{\mathbf{u}^*, h, \mathcal{P}_{q^*}(h)\}$ be a NT triple for unknown $\sigma^* = e^{2q^*}$. This implies that $\mathrm{F}(q^*,\mathbf{u}^*;h,\mathcal{P}_{q^*}(h)) = 0$. For arbitrary $q \in \mathrm{H}^1(\Omega)$, provided $q_{|\Gamma} = q^*_{|\Gamma}$, and arbitrary $\mathbf{u} \in \mathrm{W}_q(\Omega)$, we wish to interpret how $\mathrm{F}(q,\mathbf{u};h,\mathcal{P}_{q^*}(h))$ expresses errors $q-q^*$ and $\mathbf{u}-\mathbf{u}^*$. To facilitate this analysis, we write the perturbed functional as

$$\mathrm{F}(q,\mathbf{u}^*+\delta\mathbf{u};h,\mathcal{P}_{q^*}(h))\,,$$

where $\delta\mathbf{u} \in \mathrm{W}_q(\Omega)$. It is convenient to make the following definition.

DEFINITION 2.11. *Denote*

$$\mathrm{S}_q := \{\mathbf{u} \in \mathrm{W}_q(\Omega) \ : \ \mathrm{F}_{\mathrm{I}}(q,\mathbf{u}) = 0\}\,,$$

the space of F_{I}*-harmonic functions relative to* q*.*

Before addressing arbitrary $\mathbf{u} \in \mathrm{W}_q(\Omega)$, and thus arbitrary $\delta\mathbf{u} \in \mathrm{W}_q(\Omega)$, we first consider perturbations $\delta\mathbf{u}$ such that $\mathbf{u} \in \mathrm{S}_q$.

LEMMA 2.12. *Let* $\mathbf{u} \in \mathrm{S}_q$*. Then*

$$\frac{1}{c}\,\|\mathcal{P}_q(h)-\mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma} \le \mathrm{F}(q,\mathbf{u};h,\mathcal{P}_{q^*}(h)) \qquad\qquad \forall\ h \in \mathrm{H}_0^{-1/2}(\Gamma)\,,$$

where $c = 1 + \|\mathcal{P}_q\|^2_{\mathcal{L}(\mathrm{H}_0^{-1/2}(\Gamma),\mathrm{H}_0^{-1/2}(\Gamma))}$ *.*

Proof. Since $\mathbf{u} \in \mathrm{S}_q$, we immediately have that $\mathrm{F}_I(q,\mathbf{u}) = 0$. Also, writing $\mathbf{n}\cdot e^q\mathbf{u} = f$, we have that $\boldsymbol{\tau}\cdot e^{-q}\mathbf{u} = \mathcal{P}_q(f)$, and the remaining part of the functional is the boundary term:

$$\begin{aligned}\mathrm{F}_B(q,\mathbf{u};h,\mathcal{P}_{q^*}(h)) &= \|f-h\|^2_{-1/2,\Gamma} + \|\mathcal{P}_q(f)-\mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma}\\ &= \|f-h\|^2_{-1/2,\Gamma} + \|\mathcal{P}_q(f)-\mathcal{P}_q(h)+\mathcal{P}_q(h)-\mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma}\\ &= \|f-h\|^2_{-1/2,\Gamma} + \|\mathcal{P}_q(f)-\mathcal{P}_q(h)\|^2_{-1/2,\Gamma} + \|\mathcal{P}_q(h)-\mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma}\\ &\qquad + 2\langle\mathcal{P}_q(f)-\mathcal{P}_q(h),\mathcal{P}_q(h)-\mathcal{P}_{q^*}(h)\rangle_{-1/2,\Gamma}\ .\end{aligned} \tag{2.38}$$

Now, since $\mathcal{P}_q$ is linear, we have

$$\|\mathcal{P}_q\|_{\mathcal{L}(\mathrm{H}_0^{-1/2}(\Gamma),\mathrm{H}_0^{-1/2}(\Gamma))} \geq \frac{\|\mathcal{P}_q(f) - \mathcal{P}_q(h)\|_{-1/2,\Gamma}}{\|f-h\|_{-1/2,\Gamma}} \ .$$

Denoting $\zeta = \|\mathcal{P}_q\|_{\mathcal{L}(\mathrm{H}_0^{-1/2}(\Gamma),\mathrm{H}_0^{-1/2}(\Gamma))}$, we may write

$$\|f-h\|_{-1/2,\Gamma} \geq \frac{1}{\zeta}\|\mathcal{P}_q(f) - \mathcal{P}_q(h)\|_{-1/2,\Gamma} \ ,$$

so that (2.38) becomes

$$\begin{aligned}\mathrm{F}_B(q,\mathbf{u};h,\mathcal{P}_{q^*}(h)) \geq \left(1+\frac{1}{\zeta^2}\right)\|\mathcal{P}_q(f)-\mathcal{P}_q(h)\|^2_{-1/2,\Gamma} + \|\mathcal{P}_q(h)-\mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma} \\ -2\|\mathcal{P}_q(f)-\mathcal{P}_q(h)\|_{-1/2,\Gamma}\|\mathcal{P}_q(h)-\mathcal{P}_{q^*}(h)\|_{-1/2,\Gamma} \ .\end{aligned}$$

U sing an ϵ-inequality, $\forall\,\epsilon > 0$ we have

$$\begin{aligned}&\mathrm{F}_B(q,\mathbf{u};h,\mathcal{P}_{q^*}(h)) \geq \\ &\qquad\left(1+\frac{1}{\zeta^2}-\epsilon\right)\|\mathcal{P}_q(f)-\mathcal{P}_q(h)\|^2_{-1/2,\Gamma} + \left(1-\frac{1}{\epsilon}\right)\|\mathcal{P}_q(h)-\mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma} \ .\end{aligned}$$

Choosing $\epsilon = 1+\frac{1}{\zeta^2}$ leads to

$$\mathrm{F}(q,\mathbf{u};h,\mathcal{P}_{q^*}(h)) \geq \left(\frac{1}{1+\zeta^2}\right)\|\mathcal{P}_q(h)-\mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma} \ .$$

□

We now have all the necessary tools and notation to establish the main theorem, associated with the lower bound in the equivalence we seek. We write an arbitrary $\mathbf{u} \in (\mathrm{H}^1(\Omega))^2$ as $\mathbf{u}^* + \delta\mathbf{u}$ and consider its projection onto space S_q.

THEOREM 2.13. *Assume that* q^* *and* $\mathbf{u}^*$ *are such that* $\mathrm{F}(q^*,\mathbf{u}^*;h,\mathcal{P}_{q^*}(h)) = 0$, *and consider* $\mathrm{F}(q,\mathbf{u};h,\mathcal{P}_{q^*}(h))$, *where* q *and* $\mathbf{u}$ *are arbitrary insofar as* $q-q^* \in \mathrm{H}_0^1(\Omega)$ *and* $\mathbf{u} \in \mathrm{W}_q(\Omega)$. *Also, let* $\delta\mathbf{u}_s$ *satisfy*

$$\mathbf{u}^* + \delta\mathbf{u}_s = \arg\min_{\mathbf{u}^*+\delta\mathbf{u}' \in \mathrm{S}_q} \mathrm{F}(q,\mathbf{u}^*+\delta\mathbf{u}';h,\mathcal{P}_{q^*}(h)) \ , \tag{2.39}$$

and define $\delta\mathbf{u} = \mathbf{u}-\mathbf{u}^*$. *Then there exists a positive constant, c, depending on* Ω *and* $\|\mathcal{P}_q\|_{\mathcal{L}(\mathrm{H}_0^{-1/2}(\Gamma),\mathrm{H}_0^{-1/2}(\Gamma))}$, *such that*

$$\mathrm{F}(q,\mathbf{u};h,\mathcal{P}_{q^*}(h)) \geq c\left(\|\delta\mathbf{u}-\delta\mathbf{u}_s\|^2_{\mathrm{W}_q(\Omega)} + \|\mathcal{P}_q(h)-\mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma}\right) .$$

Proof. First, since $\mathbf{u}^*+\delta\mathbf{u}_s \in \mathrm{S}_q$ implies that $\mathrm{F}_I(q,\mathbf{u}^*+\delta\mathbf{u}_s) = 0$, we have

$$\begin{aligned}\mathrm{F}(q,\mathbf{u}^*+\delta\mathbf{u};h,\mathcal{P}_{q^*}(h)) &= \mathrm{F}(q,\mathbf{u}^*+\delta\mathbf{u}_s+\delta\mathbf{u}-\delta\mathbf{u}_s;h,\mathcal{P}_{q^*}(h)) \\ &= \mathrm{F}_I(q,\delta\mathbf{u}-\delta\mathbf{u}_s) + \mathrm{F}_B(q,\mathbf{u}^*+\delta\mathbf{u}_s+\delta\mathbf{u}-\delta\mathbf{u}_s;h,\mathcal{P}_{q^*}(h)) \ .\end{aligned} \tag{2.40}$$

On the boundary, we denote

$$\begin{aligned}\mathbf{n}\cdot(e^q(\mathbf{u}^*+\delta\mathbf{u}_s)) &= h_s \ , \\ \boldsymbol{\tau}\cdot(e^{-q}(\mathbf{u}^*+\delta\mathbf{u}_s)) &= \mathcal{P}_q(h_s) \ ,\end{aligned}$$

and

$$\begin{aligned} \mathbf{n} \cdot (e^{q}(\delta\mathbf{u} - \delta\mathbf{u}_s)) &= \delta h \, , \\ \boldsymbol{\tau} \cdot (e^{-q}(\delta\mathbf{u} - \delta\mathbf{u}_s)) &= \delta f \, . \end{aligned} \tag{2.41}$$

Then (2.40) becomes

$$\begin{aligned} \mathrm{F}(q, \mathbf{u}^* + \delta\mathbf{u}; h, \mathcal{P}_{q^*}(h)) = \mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s) &+ \langle h_s - h + \delta h, h_s - h + \delta h \rangle_{-1/2,\Gamma} \\ &+ \langle \mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h) + \delta f, \mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h) + \delta f \rangle_{-1/2,\Gamma} \, . \end{aligned} \tag{2.42}$$

Next, expanding the terms in (2.42) leads to

$$\begin{aligned} &\mathrm{F}(q, \mathbf{u}^* + \delta\mathbf{u}; h, \mathcal{P}_{q^*}(h)) \\ &\quad = \mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s) + \|h_s - h\|^2_{-1/2,\Gamma} + \|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma} \\ &\qquad + \|\delta h\|^2_{-1/2,\Gamma} + \|\delta f\|^2_{-1/2,\Gamma} + 2 \langle h_s - h, \delta h \rangle_{-1/2,\Gamma} + 2 \langle \mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h), \delta f \rangle_{-1/2,\Gamma} \, . \end{aligned} \tag{2.43}$$

To address the cross terms, we appeal to the orthogonality statement associated with (2.39). First, note that there exists a $\widehat{\delta\mathbf{u}}$ such that $\{\mathbf{u}^* + \widehat{\delta\mathbf{u}}, h + \delta h, \mathcal{P}_q(h + \delta h)\}$ is an NT triple. We may therefore speak of $\mathcal{P}_q(h + \delta h) = \mathcal{P}_q(h) + \mathcal{P}_q(\delta h)$, and of an orthogonality condition for $\mathbf{u}^* + \delta\mathbf{u}_s$:

$$\langle h_s - h, \delta h \rangle_{-1/2,\Gamma} + \langle \mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h), \mathcal{P}_q(\delta h) \rangle_{-1/2,\Gamma} \; = \; 0 \qquad \forall \, \delta h \, \in \, \mathrm{H}_0^{-1/2}(\Gamma) \, . \tag{2.44}$$

We can now use (2.44) to add and subtract some helpful terms so that, for any $\epsilon > 0$, (2.43) becomes

$$\begin{aligned} &\mathrm{F}(q, \mathbf{u}; h, \mathcal{P}_{q^*}(h)) \\ &\quad = \mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s) + \|h_s - h\|^2_{-1/2,\Gamma} + \|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma} + \|\delta h\|^2_{-1/2,\Gamma} \\ &\qquad + \|\delta f\|^2_{-1/2,\Gamma} + 2\epsilon \langle \mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h), \delta f - \mathcal{P}_q(\delta h) \rangle_{-1/2,\Gamma} \\ &\qquad + 2(1-\epsilon) \langle h_s - h, \delta h \rangle_{-1/2,\Gamma} + 2(1-\epsilon) \langle \mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h), \delta f \rangle_{-1/2,\Gamma} \\ &\quad \geq \mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s) + \|h_s - h\|^2_{-1/2,\Gamma} + \|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma} + \|\delta h\|^2_{-1/2,\Gamma} \\ &\qquad + \|\delta f\|^2_{-1/2,\Gamma} - 2\epsilon \|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|_{-1/2,\Gamma} \|\delta f - \mathcal{P}_q(\delta h)\|_{-1/2,\Gamma} \\ &\qquad - 2(1-\epsilon) \left| \langle h_s - h, \delta h \rangle_{-1/2,\Gamma} \right| - 2(1-\epsilon) \left| \langle \mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h), \delta f \rangle_{-1/2,\Gamma} \right| \, . \end{aligned} \tag{2.45}$$

Focusing on the $(1-\epsilon)$ terms, by Cauchy-Schwarz we have

$$\begin{aligned} 2 \left| \langle h_s - h, \delta h \rangle_{-1/2,\Gamma} \right| &\leq \|h_s - h\|^2_{-1/2,\Gamma} + \|\delta h\|^2_{-1/2,\Gamma} \\ 2 \left| \langle \mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h), \delta f \rangle_{-1/2,\Gamma} \right| &\leq \|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma} + \|\delta f\|^2_{-1/2,\Gamma} \, . \end{aligned}$$

Applying these bounds to (2.45), we have

$$\begin{aligned} &\mathrm{F}(q, \mathbf{u}; h, \mathcal{P}_{q^*}(h)) \\ &\quad \geq \mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s) - 2\epsilon \|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|_{-1/2,\Gamma} \|\delta f - \mathcal{P}_q(\delta h)\|_{-1/2,\Gamma} \\ &\qquad + \epsilon \left(\|h_s - h\|^2_{-1/2,\Gamma} + \|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma} + \|\delta h\|^2_{-1/2,\Gamma} + \|\delta f\|^2_{-1/2,\Gamma} \right) \, . \end{aligned} \tag{2.46}$$

Now, as a direct application of Lemma 2.10, there exists a constant c_1 such that

$$\|\delta f - \mathcal{P}_q(\delta h)\|^2_{-1/2,\Gamma} \le c_1 \,\mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s)\,. \tag{2.47}$$

To see this, first note that $\exists\, \tilde{\mathbf{u}} \in \mathrm{W}_q(\Omega)$ such that $\{\tilde{\mathbf{u}}, h_s + \delta h, \mathcal{P}_q(h_s) + \mathcal{P}_q(\delta h)\}$ is an NT triple for q. Then, observe that if we let $\mathbf{w}_h = \tilde{\mathbf{u}}$ and $\mathbf{w} = \mathbf{u}^* + \delta\mathbf{u}_s + \delta\mathbf{u} - \delta\mathbf{u}_s$, we may write

$$\boldsymbol{\tau} \cdot (e^{-q}(\mathbf{w} - \mathbf{w}_h)) = \delta f - \mathcal{P}_q(\delta h)\;. \tag{2.48}$$

Therefore, applying Lemma 2.10, we have

$$\begin{aligned} \|\delta f - \mathcal{P}_q(\delta h)\|^2_{-1/2,\Gamma} &\le c\,\mathrm{F}_I(q, \mathbf{w} - \mathbf{w}_h) \\ &\le c\,\mathrm{F}_I(q, \mathbf{w}) \\ &\le c_1\,\mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s)\,. \end{aligned} \tag{2.49}$$

As a direct result of (2.49), we may write (2.46) as

$$\begin{aligned} \mathrm{F}(q, \mathbf{u}; h, \mathcal{P}_{q^*}(h)) \ge \mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s) &+ \epsilon\|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma} \\ &- 2\epsilon c_1 \|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|_{-1/2,\Gamma}\left(\mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s)\right)^{1/2} \\ &+ \epsilon\left(\|h_s - h\|^2_{-1/2,\Gamma} + \|\delta h\|^2_{-1/2,\Gamma} + \|\delta f\|^2_{-1/2,\Gamma}\right). \end{aligned} \tag{2.50}$$

Next, we use an ϵ-inequality, with η in place of ϵ, to rewrite the first three terms: For any $\eta > 0$,

$$\begin{aligned} -2\epsilon c_1 &\|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|_{-1/2,\Gamma}(\mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s))^{1/2} \\ &\ge -\epsilon c_1^2 \eta \mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s) - \frac{\epsilon}{\eta}\|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma}\,, \end{aligned} \tag{2.51}$$

and (2.50) becomes

$$\begin{aligned} \mathrm{F}(q, \mathbf{u}; h, \mathcal{P}_{q^*}(h)) \ge (1 - \epsilon c_1^2\eta)\mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s) &+ \epsilon\left(1 - \frac{1}{\eta}\right)\|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma} \\ &+ \epsilon\left(\|h_s - h\|^2_{-1/2,\Gamma} + \|\delta h\|^2_{-1/2,\Gamma} + \|\delta f\|^2_{-1/2,\Gamma}\right)\;. \end{aligned}$$

Choosing $0 < \epsilon < 1$ and $\eta > 1$ so that $(1 - \epsilon c_1^2\eta) = \epsilon(1 - 1/\eta)$ and ϵ is as large as possible, we have

$$\begin{aligned} \mathrm{F}(q, \mathbf{u}; h, \mathcal{P}_{q^*}(h)) \ge \epsilon\left(1 - \frac{1}{\eta}\right)&\left(\mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s) + \|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma}\right) \\ &+ \epsilon\left(\|\delta h\|^2_{-1/2,\Gamma} + \|\delta f\|^2_{-1/2,\Gamma} + \|h_s - h\|^2_{-1/2,\Gamma}\right)\;. \end{aligned} \tag{2.52}$$

Note that this is always possible since $\epsilon = \left(c_1^2\eta + (1 - 1/\eta)\right)^{-1}$. Now, we use

$$\mathrm{F}(q, \delta\mathbf{u} - \delta\mathbf{u}_s; 0, 0) = \mathrm{F}_I(q, \delta\mathbf{u} - \delta\mathbf{u}_s) + \|\delta h\|^2_{-1/2,\Gamma} + \|\delta f\|^2_{-1/2,\Gamma}$$

so that we may write

$$\begin{aligned} &\mathrm{F}(q, \mathbf{u}; h, \mathcal{P}_{q^*}(h)) \\ &\qquad \ge c_2\left(\mathrm{F}(q, \delta\mathbf{u} - \delta\mathbf{u}_s; 0, 0) + \|h_s - h\|^2_{-1/2,\Gamma} + \|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma}\right)\,, \end{aligned}$$

where $c_2 := \epsilon(1 - 1/\eta)$. Next, we recall that

$$\begin{aligned}\mathrm{F}(q, \mathbf{u}^* + \delta\mathbf{u}_s; h, \mathcal{P}_{q^*}(h)) &= \mathrm{F}_B(q, \mathbf{u}^* + \delta\mathbf{u}_s; h, \mathcal{P}_{q^*}(h)) \\ &= \|h_s - h\|^2_{-1/2,\Gamma} + \|\mathcal{P}_q(h_s) - \mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma} ,\end{aligned}$$

since $\mathbf{u}^* + \delta\mathbf{u}_s \in \mathrm{S}_q$. Therefore, we may apply Lemma 2.12 to arrive at

$$\mathrm{F}(q, \mathbf{u}; h, \mathcal{P}_{q^*}(h)) \ \geq \ c\left(\mathrm{F}(q, \delta\mathbf{u} - \delta\mathbf{u}_s; 0, 0) + \|\mathcal{P}_q(h) - \mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma}\right) .$$

Finally, by Lemma 2.9, we have the desired coercive bound. □

We now state an equivalence result for a fixed NT pair, $\{h, \mathcal{P}_{q^*}(h)\}$, which immediately follows from Theorem 2.13 and a simple derivation of the upper bound.

COROLLARY 2.14. *Let $h \in \mathrm{H}_0^{-1/2}(\Gamma)$ and recall that $\sigma = e^{2q}$ and $\sigma^* = e^{2q^*}$. Then there exists a positive constant, c, depending only on Ω and $\|\mathcal{P}_q\|_{\mathcal{L}(\mathrm{H}_0^{-1/2}(\Gamma),\mathrm{H}_0^{-1/2}(\Gamma))}$, such that*

$$\begin{aligned}c\|\mathcal{R}_\sigma(h) - \mathcal{R}_{\sigma^*}(h)\|^2_{1/2,\Gamma} & \\ \leq \min_{\mathbf{u}\,\epsilon\,\mathrm{W}_q(\Omega)} \mathrm{F}(q, \mathbf{u}; h, \mathcal{P}_{q^*}(h)) & \\ \leq \|\mathcal{R}_\sigma(h) - \mathcal{R}_{\sigma^*}(h)\|^2_{1/2,\Gamma} . & \qquad (2.53)\end{aligned}$$

Proof. The lower bound follows immediately from applying Lemma 2.8 to the statement in Theorem 2.13 and performing the minimization in (2.23). To show continuity, we simply restrict the space over which the minimization is conducted and use the definition of S_q to see that

$$\begin{aligned}\min_{\mathbf{u}\,\epsilon\,\mathrm{W}_q(\Omega)} \mathrm{F}(q, \mathbf{u}; h, \mathcal{P}_{q^*}(h)) \ &\leq \ \min_{\mathbf{u}\in\mathrm{S}_q} \mathrm{F}(q, \mathbf{u}; h, \mathcal{P}_{q^*}(h)) \\ &\leq \ \min_{\{\mathbf{u}\in\mathrm{S}_q:\mathbf{n}\cdot\mathbf{u}=h\}} \mathrm{F}_B(q, \mathbf{u}; h, \mathcal{P}_{q^*}(h)) \\ &= \ \|\mathcal{P}_q(h) - \mathcal{P}_{q^*}(h)\|^2_{-1/2,\Gamma} , \qquad (2.54)\end{aligned}$$

for any $h \in \mathrm{H}_0^{-1/2}(\Gamma)$. Hence, again appealing to Lemma 2.8 to relate $\mathcal{P}$ to $\mathcal{R}$ proves the upper bound in (2.53). □

The above corollary immediately leads to the equivalence of the OLS and FOSLS functionals, which we now state formally.

COROLLARY 2.15. *With $\mathcal{F}_{OSLS}$ defined in (2.23), there exists a positive constant, c, depending only on Ω and $\|\mathcal{P}_q\|_{\mathcal{L}(\mathrm{H}_0^{-1/2}(\Gamma),\mathrm{H}_0^{-1/2}(\Gamma))}$, such that*

$$\begin{aligned}c\sum_{i=1}^{L}\|\mathcal{R}_\sigma(h_i) - \mathcal{R}_{\sigma^*}(h_i)\|^2_{1/2,\Gamma} & \\ \leq \mathcal{F}_{OSLS}(q; \{h_i, \mathcal{P}_{q^*}(h_i)\}_{i=1}^{L}) & \\ \leq \sum_{i=1}^{L}\|\mathcal{R}_\sigma(h_i) - \mathcal{R}_{\sigma^*}(h_i)\|^2_{1/2,\Gamma} . & \qquad (2.55)\end{aligned}$$

As in the previous section, we can also state the equivalence of a sup version of the FOSLS functional and the operator norm of the defect in the ND maps. For $c > 0$

depending only on Ω and $\|\mathcal{P}_q\|_{\mathcal{L}(\mathrm{H}_0^{-1/2}(\Gamma),\mathrm{H}_0^{-1/2}(\Gamma))}$, we may write

$$
\begin{aligned}
c\|\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*}\|^2_{\mathcal{L}(\mathrm{H}_0^{-1/2}(\Gamma),\mathrm{H}_0^{1/2}(\Gamma))} \\
\leq \quad \sup_{h\in \mathrm{H}_0^{-1/2}(\Gamma)} \frac{\left(\min_{\mathbf{u}\,\epsilon\,\mathrm{W}_q(\Omega)} \mathrm{F}(q,\mathbf{u};h,\mathcal{P}_{q^*}(h))\right)}{\|h\|_{-1/2,\Gamma}} \\
\leq \|\mathcal{R}_\sigma - \mathcal{R}_{\sigma^*}\|^2_{\mathcal{L}(\mathrm{H}_0^{-1/2}(\Gamma),\mathrm{H}_0^{1/2}(\Gamma))}\,.
\end{aligned}
$$

3. Unifying FOSLS framework. We have established the equivalence of various least-squares methods, as well as alluded to the balance in weighting the boundary and interior functionals. Another connection between the Kohn and Vogelius and FOSLS functionals further reveals the underlying framework. Considering a single DN pair, the FOSLS interior functional may be written as

$$
\left\| \begin{pmatrix} e^{-q}\nabla\cdot e^{q} \\ e^{q}\nabla\times e^{-q} \end{pmatrix} \mathbf{u} \right\|^2_{0,\Omega} := \|\mathcal{L}\mathbf{u}\|^2_{0,\Omega}\,.
$$

Alternatively, one can arrive at the Kohn and Vogelius functional using a generalized Helmholtz decomposition for $\mathbf{u}$, leading to

$$
\begin{aligned}
\mathbf{u} &= e^{q}\nabla e^{-q} r - e^{-q}\nabla^{\perp} e^{q} t \\
&= \mathcal{L}^* \begin{pmatrix} r \\ t \end{pmatrix}\,,
\end{aligned}
$$

for some $r,t \in \mathrm{H}^1(\Omega)$. The Kohn and Vogelius functional in this context, for a single DN pair, may be written as

$$
\|e^{q}\nabla p - e^{-q}\nabla^{\perp} s\|^2_{0,\Omega} = \left\| \mathcal{L}^* \begin{pmatrix} e^{q}p \\ e^{-q}s \end{pmatrix} \right\|^2_{0,\Omega}\,,
$$

and may thus be viewed as a first-order system LL* (FOSLL*) formulation [7]. Indeed, in being adjoint formulations, the $\mathcal{L}$ in FOSLS has over-specified boundary conditions, while the corresponding $\mathcal{L}^*$ has no boundary conditions. FOSLL* is in its formative stages as a viable methodology for the solution of PDEs, yet it may prove particularly effective in the forward solution of Maxwell's equations, for which regularity requirements of standard FOSLS may be too stringent. As such, the Kohn and Vogelius formulation represents an interesting connection between FOSLL* and the inverse problem.

The stage is thus set for [27], wherein we develop the sense in which each least-squares formulation within the FOSLS framework may be posed well. Naturally, we cater the development there to Output Least Squares, relying on the equivalence statements established above to bring the ensuing theory to bear on each formulation within the framework.

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