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FIRST-ORDER SYSTEM LEAST SQUARES (FOSLS) FOR SPATIAL LINEAR ELASTICITY: PURE TRACTION*

SANG DONG KIM†, THOMAS A. MANTEUFFEL‡, AND STEPHEN F. MCCORMICK‡

Abstract. This paper develops first-order system least-squares (FOSLS) functionals for solving the pure traction problem in three-dimensional linear elasticity. It is a direct extension of an earlier paper on planar elasticity [Z. Cai, T. A. Manteuffel, S. F. McCormick, and S. V. Parter, *SIAM J. Numer. Anal.*, 35 (1998), pp. 320–335]. Two functionals are developed, one involving L^2 norms of the first-order system and the other involving dual norms. These functionals are shown to be equivalent to appropriate product Sobolev norms, *uniformly* in the Poisson ratio. These results imply that standard finite element discretization and iterative solver techniques can be applied to obtain performance that is optimal—even as the material nears the incompressible limit.

Key words. least-squares discretization, linear elasticity, pure traction, Poisson ratio

AMS subject classifications. 65F10, 65F30

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1. Introduction.

The first-order system least-squares (FOSLS) approach for the pure traction problem in planar linear elasticity was introduced in [8]. The focus there was on two FOSLS functionals, one involving only L^2 norms and the other incorporating dual norms (H^{-1} norms). Under certain regularity assumptions (H^2 and H^1, respectively) for elasticity *and* related Stokes equations, *equivalence* (i.e., continuity and coercivity) was established between the homogeneous parts of these functionals and appropriate product Sobolev norms. The goal of the current paper is to provide an extension to three dimensions of these earlier results.

The extension of the dual-norm functional to three dimensions involves a functional with nonstandard dual norms and the restriction of the domain to be simply connected. With these restrictions, standard variational arguments are used to establish equivalence. Under assumptions that imply H^2 regularity, the assumption of a simply connected domain can be removed and H^{-1}-like norms can be employed.

Our proof of equivalence for the L^2 functional requires full H^2 regularity and depends on establishing regularity for a large Stokes-like system with nonstandard boundary conditions. For this proof we have appealed to Agmon–Douglis–Nirenberg (ADN) theory [1].

We refer the reader to [8, 5] for background and discussion of some practical issues, but we will include the necessary technical details here for completeness. The problem, notation, and other definitions are given in the next section. Norm equivalence for both the L^2 and the dual-norm FOSLS functionals are developed in section 3, along with some remarks on the practical implementation of the dual-norm functional. Section 4 provides a rigorous proof of regularity for the associated Stokes-like

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†Mathematics Department, Kyungpook National University, Taegu 702-701, Korea (skim@sobolev.kyungpook.ac.kr). The work of this author was sponsored by KOSEF 1999-2-103-002-3.

‡Department of Applied Mathematics, Campus Box 526, University of Colorado at Boulder, Boulder, CO 80309-0526 (tmanteuf@boulder.colorado.edu, stevem@boulder.colorado.edu). The work of these authors was sponsored by National Science Foundation grant DMS-9706866, Department of Energy grant DE-FG03-93ER25217, and National Eye Institute grant R01-EY12291-01.

equations. We apologize to the reader for the length of this section, but we could not find a shorter proof.

2. The elasticity problem, its first-order system formulations, and other preliminaries. Let Ω be a bounded, open, connected domain in $\Re^3$ with Lipschitz boundary $\partial\Omega$. Suppose that $\partial\Omega$ consists of a finite number of disjoint simple closed surfaces, Γ_i, $i = 0, 1, \ldots, m$. Denote the Lamé constants by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)},$$

where $E > 0$ is the *modulus of elasticity* and $\nu = \frac{\lambda}{2(\lambda+\mu)}$ is the *Poisson ratio* of the elastic material. We assume that the elastic material is isotropic, homogeneous, and strongly elliptic, so that $\mu > 0$ and $3\lambda + 2\mu > 0$ $(-1 < \nu < 1/2)$ (cf. [15]). We write the system of equations of linear elasticity for the *displacement* $\mathbf{u} = (u_1, u_2, u_3)^t$, with *pure traction* boundary conditions, as follows (cf. [9]):

$$\left\{ \begin{array}{rcl} -\mu\Delta\,\mathbf{u} - (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} &=& \mathbf{f} \quad \text{in} \quad \Omega, \\ \sum_{j=1}^{3} \sigma_{ij}(\mathbf{u})\, n_j &=& 0 \quad \text{on} \quad \partial\Omega, \quad 1 \le i \le 3, \end{array} \right. \tag{2.1}$$

where the symbols Δ, ∇, and $\nabla\cdot$ stand for the Laplacian, gradient, and divergence operators, respectively ($\Delta\mathbf{u}$ is the vector of components $\Delta\, u_i$); $\mathbf{f}$ is a given vector function; $\sigma_{ij}(\mathbf{u}) = \lambda(\nabla\cdot\mathbf{u})\delta_{ij} + 2\mu\epsilon_{ij}(\mathbf{u})$ is the *stress tensor*, $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ is the *deformation tensor*, and δ_{ij} is the Kronecker delta symbol; and $\mathbf{n} = (n_1, n_2, n_3)^t$ is the outward unit normal on the boundary.

We will be introducing a new independent variable related to the n^2-vector function of gradients of the displacement vectors, u_i, $i = 1, 2, 3$. For the first three sections of this paper, it is convenient to view the original n-vector functions as column vectors and the new n^2-vector functions as block column vectors. Thus, given

$$\mathbf{u} = (u_1, u_2, u_3)^t,$$

then an operator G defined on scalar functions is extended to n-vectors componentwise:

$$G\mathbf{u} = \begin{pmatrix} Gu_1 \\ Gu_2 \\ \vdots \\ Gu_n \end{pmatrix}.$$

For example,

$$\nabla\mathbf{u} = (\partial_1 u_1, \partial_2 u_1, \partial_3 u_1, \partial_1 u_2, \partial_2 u_2, \partial_3 u_2, \partial_1 u_3, \partial_2 u_3, \partial_3 u_3)^t.$$

If $\mathbf{U}_i \equiv G\,u_i$ is a vector function, then we write the block column vector

$$\mathbf{U} \equiv G\mathbf{u} = \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \vdots \\ \mathbf{U}_n \end{pmatrix}.$$

If D is an operator on vector functions, then its extension to block column vectors is defined by

$$D\mathbf{U} = \begin{pmatrix} D\mathbf{U}_1 \\ D\mathbf{U}_2 \\ \vdots \\ D\mathbf{U}_n \end{pmatrix}.$$

For example, for $\mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3)^t$ where $\mathbf{U}_i = (U_{i1}, U_{i2}, U_{i3})^t$, then

$$\nabla \times \mathbf{U} = \begin{pmatrix} \nabla \times \mathbf{U}_1 \\ \nabla \times \mathbf{U}_2 \\ \nabla \times \mathbf{U}_3 \end{pmatrix} = \begin{array}{c} (-\partial_3 U_{12} + \partial_2 U_{13},\, \partial_3 U_{11} - \partial_1 U_{13},\, -\partial_2 U_{11} + \partial_1 U_{12}, \\ -\partial_3 U_{22} + \partial_2 U_{23},\, \partial_3 U_{21} - \partial_1 U_{23},\, -\partial_2 U_{21} + \partial_1 U_{22}, \\ -\partial_3 U_{32} + \partial_2 U_{33},\, \partial_3 U_{31} - \partial_1 U_{33},\, -\partial_2 U_{31} + \partial_1 U_{32})^t \end{array}$$

and

$$\nabla \cdot \mathbf{U} = (\nabla \cdot \mathbf{U}_1, \nabla \cdot \mathbf{U}_2, \nabla \cdot \mathbf{U}_3)^t.$$

We also extend the respective normal and tangential operators $\mathbf{n}\cdot$ and $\mathbf{n}\times$ componentwise:

$$\mathbf{n} \cdot \mathbf{U} = \begin{pmatrix} \mathbf{n} \cdot \mathbf{U}_1 \\ \mathbf{n} \cdot \mathbf{U}_2 \\ \vdots \\ \mathbf{n} \cdot \mathbf{U}_n \end{pmatrix} \quad \text{and} \quad \mathbf{n} \times \mathbf{U} = \begin{pmatrix} \mathbf{n} \times \mathbf{U}_1 \\ \mathbf{n} \times \mathbf{U}_2 \\ \vdots \\ \mathbf{n} \times \mathbf{U}_n \end{pmatrix}.$$

The inner products and norms on the block column vector functions are defined in the natural componentwise way:

$$\|\mathbf{U}\|^2 = \sum_{i=1}^{n} \|\mathbf{U}_i\|^2.$$

Define the trace operator tr by

$$\mathrm{tr}\mathbf{U} = \sum_{i=1}^{n} U_{i,i}.$$

With the exception of (2.4), we denote vectors of length 3 by lowercase bold letters and vectors of length 9 by uppercase bold letters. For the last section only, it will be more convenient to view the n^2-vector functions as matrices. Thus, for example, we will write

$$\mathbf{U}_{3\times 3} = (\nabla \mathbf{u}_1, \nabla \mathbf{u}_2, \nabla \mathbf{u}_3)_{3\times 3} = (\partial_i u_j)_{3\times 3}.$$

We will frequently use subscripts there to avoid confusion. Thus, for example, we will denote the 3×3 identity matrix by I_3. We will also make frequent use of standard tensor product notation.

We use standard notation and definitions for the Sobolev spaces $H^s(\Omega)^d$, associated inner products $\langle \cdot, \cdot \rangle_s$, and respective norms $\| \cdot \|_s$, $s \geq 0$. (We suppress the designations d and Ω on the inner products and norms when dependence on dimension and region is clear by context.) $H^0(\Omega)^d$ coincides with $L^2(\Omega)^d$, in which case

the norm and inner product will be denoted by $\|\cdot\|$ and $\langle\cdot,\,\cdot\rangle$, respectively. As usual, $H_0^s(\Omega)$ will denote the closure with respect to the norm $\|\cdot\|_s$ of the space of infinitely differentiable functions with compact support in Ω, and $L_0^2(\Omega)$ will denote the space of $L^2(\Omega)$ functions p such that $\int_\Omega p dx = 0$.

We use $H_0^{-1}(\Omega)$ and $H^{-1}(\Omega)$ to denote the dual spaces of $H_0^1(\Omega)$ and $H^1(\Omega)/\Re$ with norms defined by

$$\|\phi\|_{-1,\,0} = \sup_{\psi\in H_0^1(\Omega)} \frac{\langle\phi,\,\psi\rangle}{|\psi|_1}$$

and

$$\|\phi\|_{-1} = \sup_{\psi\in H^1(\Omega)/\Re} \frac{\langle\phi,\,\psi\rangle}{|\psi|_1},$$

respectively.

Let $\mathcal{N}$ denote the space of *infinitesimal rigid motions* [10]:

$$\mathcal{N} = \{(a_1 - b_3x_2 + b_2x_3,\, a_2 + b_3x_1 - b_1x_3,\, a_3 - b_2x_1 + b_1x_2)^t \,:\, a_i, b_i \in \Re\}.$$

Let $\mathcal{N}^\perp$ denote its orthogonal complement in $L^2(\Omega)^3$. Following [4], let $\mathcal{N}^c$ denote its complement in $H^1(\Omega)^3$ defined so that $\mathbf{u}\in\mathcal{N}^c$ if and only if

$$\int_\Omega u_1\,dx = \int_\Omega u_2\,dx = \int_\Omega u_3\,dx = 0 \tag{2.2}$$

and

$$\int_\Omega (\partial_2u_1 - \partial_1u_2)\,dx = \int_\Omega (\partial_3u_2 - \partial_2u_3)\,dx = \int_\Omega (\partial_3u_1 - \partial_1u_3)\,dx = 0. \tag{2.3}$$

Then (the weak form of) boundary value problem (2.1) has a unique solution $\mathbf{u}\in H^1(\Omega)^3\cap\mathcal{N}^c$ for any $\mathbf{f}\in L^2(\Omega)^3\cap\mathcal{N}^\perp$ (cf. [10]).

Let $A = \lambda A_1 + 2\mu A_2$ be the 9×9 matrix defined as follows:

$$\begin{aligned} A_1 &= \mathbf{b}\,\mathbf{b}^t,\\ \mathbf{b} &= (1,\,0,\,0,\;\,0,\,1,\,0,\;\,0,\,0,\,1)^t, \end{aligned} \tag{2.4}$$

and

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0\\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $\mathbf{U} \in \text{Range}(A)$ if and only if $\mathbf{U}_{ij} = \mathbf{U}_{ji}$, $1 \le i, j \le 3$. We will call $\mathbf{U}$ symmetric if this holds. Moreover, $\mathbf{N} \in \text{Null}(A)$ if and only if $\mathbf{N}_{ij} = -\mathbf{N}_{ji}$, $1 \le i, j \le 3$. Then the elasticity equations in (2.1) may be rewritten in the compact form

$$\left\{ \begin{array}{rcll} -\nabla \cdot A\nabla \mathbf{u} & = & \mathbf{f} & \text{in} \quad \Omega, \\ \mathbf{n} \cdot (A\nabla \mathbf{u}) & = & \mathbf{0} & \text{on} \quad \partial\Omega. \end{array} \right. \tag{2.5}$$

We introduce the *displacement flux* variable $\mathbf{U} = \nabla \mathbf{u}$:

$$\begin{aligned} \mathbf{U} &= (U_{11}, U_{12}, U_{13}, U_{21}, U_{22}, U_{23}, U_{31}, U_{32}, U_{33})^t \\ &= (\partial_1 u_1, \partial_2 u_1, \partial_3 u_1, \partial_1 u_2, \partial_2 u_2, \partial_3 u_2, \partial_1 u_3, \partial_2 u_3, \partial_3 u_3)^t. \end{aligned}$$

Since the definition of $\mathbf{U}$ implies that

$$\nabla \times \mathbf{U} = \mathbf{0} \quad \text{in} \quad \Omega, \tag{2.6}$$

then a system that is equivalent to (2.5) is

$$\left\{ \begin{array}{rcll} \mathbf{U} - \nabla \mathbf{u} & = & \mathbf{0} & \text{in} \quad \Omega, \\ -\nabla \cdot A\mathbf{U} & = & \mathbf{f} & \text{in} \quad \Omega, \\ \nabla \times \mathbf{U} & = & \mathbf{0} & \text{in} \quad \Omega, \\ \mathbf{n} \cdot A\mathbf{U} & = & \mathbf{0} & \text{on} \quad \partial\Omega. \end{array} \right. \tag{2.7}$$

Of perhaps more practical importance is the system that involves $\mathbf{U}$ only:

$$\left\{ \begin{array}{rcll} -\nabla \cdot A\mathbf{U} & = & \mathbf{f} & \text{in} \quad \Omega, \\ \nabla \times \mathbf{U} & = & \mathbf{0} & \text{in} \quad \Omega, \\ \mathbf{n} \cdot A\mathbf{U} & = & \mathbf{0} & \text{on} \quad \partial\Omega. \end{array} \right. \tag{2.8}$$

We will show in section 3 that this reduced system is well posed and that it is perhaps better suited to FOSLS treatment, especially near the incompressible limit, that is, when $\lambda \to \infty$ (cf. [8, section 1]).

Define the solution space for the primitive variables by

$$\mathcal{W} = \{\mathbf{u} \in H^1(\Omega)^3 \cap \mathcal{N}^c : \nabla \cdot A\nabla \mathbf{u} \in L^2(\Omega)^3, \, \mathbf{n} \cdot A\nabla \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega\}$$

and

$$\mathcal{Y} = \{\mathbf{u} \in H^1(\Omega)^3 \cap \mathcal{N}^c\}.$$

We have posed (2.1) on the space for which $\nabla \mathbf{u}$ is orthogonal in $L^2(\Omega)^9$ to gradients of elements of $\mathcal{N}$ (we write $\nabla \mathbf{u} \in (\nabla \mathcal{N})^\perp$), so we are at liberty to impose the condition (2.3), that is, $\mathbf{U} \in (\nabla \mathcal{N})^\perp$:

$$\int_\Omega (U_{21} - U_{12}) \, dx = \int_\Omega (U_{13} - U_{31}) \, dx = \int_\Omega (U_{32} - U_{23}) \, dx = 0.$$

We thus define the solution space for the new variables by

$$\mathcal{U} = \{\mathbf{U} \in L^2(\Omega)^9 \cap (\nabla \mathcal{N})^\perp \, : \, \nabla \cdot A\mathbf{U} \in L^2(\Omega)^3, \, \mathbf{n} \cdot A\mathbf{U} = \mathbf{0} \text{ on } \partial\Omega\}$$

for the case of general domain Ω and by

$$\mathcal{V} = \{\mathbf{U} \in H^1(\Omega)^9 \cap (\nabla \mathcal{N})^\perp \, : \, \mathbf{n} \cdot A\mathbf{U} = \mathbf{0} \text{ on } \partial\Omega\}$$

for the case that domain Ω is a convex polyhedron or has $C^{1,1}$ boundary. In the context of L^2 norms, $\mathbf{u} \in \mathcal{W}$ is a natural choice for (2.5) and $\mathbf{u} \in \mathcal{Y}$ is a natural choice for (2.7). Our theory will show that $\mathbf{U} \in \mathcal{U}$ is a natural choice for (2.8) in the context of H^{-1} norms and that $\mathbf{U} \in \mathcal{V}$ is a natural choice for (2.7) or (2.8) in the context of L^2 norms.

2.1. Dual norms and spaces. We now define the dual norms that we will use to construct a dual-norm functional. In what follows, C, with or without subscripts, will be used to represent a generic constant that depends only on the domain Ω. The norms dual to the spaces $(H_0^1(\Omega))^3$ and $(H^1(\Omega)/\Re)^3$ are defined by

$$\|\mathbf{u}\|_{-1,\,0} := \sup_{\mathbf{w}\in(H_0^1(\Omega))^3} \frac{\langle \mathbf{u},\, \mathbf{w}\rangle}{|\mathbf{w}|_1} \tag{2.9}$$

and

$$\|\mathbf{u}\|_{-1} := \sup_{\mathbf{w}\in(H^1(\Omega)/\Re)^3} \frac{\langle \mathbf{u},\, \mathbf{w}\rangle}{|\mathbf{w}|_1}, \tag{2.10}$$

respectively. We also consider the spaces

$$H(\mathrm{div}) := \{\mathbf{u} \in (L^2(\Omega))^3 \ : \ \nabla\cdot\mathbf{u} \in (L^2(\Omega))^3\}, \tag{2.11}$$

$$H_0(\mathrm{div}) := \{\mathbf{u} \in H(\mathrm{div}) \ : \ \mathbf{n}\cdot\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega\}, \tag{2.12}$$

$$H(\mathrm{curl}) := \{\mathbf{u} \in (L^2(\Omega))^3 \ : \ \nabla\times\mathbf{u} \in (L^2(\Omega))^3\}, \tag{2.13}$$

$$H_0(\mathrm{curl}) := \{\mathbf{u} \in H(\mathrm{curl}) \ : \ \mathbf{n}\times\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega\}. \tag{2.14}$$

Two important subspaces of $(L^2(\Omega))^3$ are

$$D_0C := H_0(\mathrm{div}) \cap H(\mathrm{curl}), \tag{2.15}$$

$$DC_0 := H(\mathrm{div}) \cap H_0(\mathrm{curl}), \tag{2.16}$$

with seminorm, norm, and inner product

$$\begin{aligned} |\mathbf{u}|_{DC}^2 &:= \langle \nabla\cdot\mathbf{u},\, \nabla\cdot\mathbf{u}\rangle + \langle \nabla\times\mathbf{u},\, \nabla\times\mathbf{u}\rangle, \\ \|\mathbf{u}\|_{DC}^2 &:= |\mathbf{u}|_{DC}^2 + \|\mathbf{u}\|^2, \\ \langle \mathbf{u},\, \mathbf{v}\rangle_{DC} &:= \langle \nabla\cdot\mathbf{u},\, \nabla\cdot\mathbf{v}\rangle + \langle \nabla\times\mathbf{u},\, \nabla\times\mathbf{v}\rangle + \langle \mathbf{u},\, \mathbf{v}\rangle. \end{aligned}$$

The following result for simply connected Ω is found in [11].

LEMMA 2.1. *If Ω is simply connected, then there exists C depending only on Ω such that*

$$\|\mathbf{u}\| \le C(\|\nabla\cdot\mathbf{u}\| + \|\nabla\times\mathbf{u}\|) \qquad \forall\ \mathbf{u} \in D_0C. \tag{2.17}$$

If, in addition, $\partial\Omega$ is connected, then there exists C depending only on Ω such that (2.17) *holds for every $\mathbf{u} \in DC_0$.*

Proof. For the proof, see [11, Chapter I, section 3.4, Lemma 3.4, and section 3.5, Lemma 3.6]. ∎

If Ω is simply connected and $\partial\Omega$ has $m+1$ components, Γ_i, $i = 0, \ldots, m$, then there is a subspace $Q \subset DC_0$ of dimension m for which $\mathbf{u} \in Q$ implies $|\mathbf{u}|_{DC} = 0$. This subspace has basis $\{\boldsymbol{\phi}_i\}_{i=1}^m$, where $\boldsymbol{\phi}_i = \nabla\zeta_i$ and ζ_i satisfies

$$\left\{ \begin{array}{rcll} \Delta\zeta_i &=& 0 & \text{in} \quad \Omega, \\ \zeta_i &=& 1 & \text{on} \quad \Gamma_i, \\ \zeta_i &=& 0 & \text{on} \quad \Gamma_j, \quad j \ne i. \end{array} \right. \tag{2.18}$$

The quotient space DC_0/Q is characterized by

$$DC_0/Q := \left\{ \mathbf{u} \in DC_0 \ : \ \int_{\Gamma_i} \mathbf{n}\cdot\mathbf{u} = \langle \mathbf{n}\cdot\mathbf{u},\, 1\rangle_{\Gamma_i} = 0, \quad i = 1, \ldots, m \right\}. \tag{2.19}$$

Here, $\langle \cdot, \cdot \rangle_{\Gamma_i}$ represents the duality pairing between $H^{1/2}(\Gamma_i)$ and $H^{-1/2}(\Gamma_i)$ (cf. [11]). We also define

$$\begin{aligned} D_f &:= \{\mathbf{u} \in H_0(\text{div}) \ : \ \nabla \cdot \mathbf{u} = 0\}, \\ C_f &:= \{\mathbf{u} \in H_0(\text{curl}) \ : \ \nabla \times \mathbf{u} = 0\}, \\ \mathcal{L} &:= \{\mathbf{u} \in DC_0 \ : \ \nabla \cdot \mathbf{u} = 0, \langle \mathbf{n} \cdot \mathbf{u}, 1 \rangle_{\Gamma_i} = 0, \quad i = 1, \ldots, m\}. \end{aligned}$$

Notice that $\mathcal{L} \subset DC_0/Q$ is a Hilbert space under the DC norm.

LEMMA 2.2. *If Ω is simply connected, then there exists C_1 and C_2 depending only on Ω such that*

$$\|\mathbf{u}\| \le C_1 \|\nabla \times \mathbf{u}\| \qquad \forall \ \mathbf{u} \in \mathcal{L}, \tag{2.20}$$

$$\|\mathbf{u}\| \le C_2 \left(\|\nabla \times \mathbf{u}\| + \|\nabla \cdot \mathbf{u}\|\right) \qquad \forall \ \mathbf{u} \in DC_0/Q. \tag{2.21}$$

Proof. The first result follows from [11, Chapter I, Theorem 3.6 and the proof of Lemma 3.4]. Theorem 3.6 states that the curl is a bijection from $\mathcal{L}$ to D_f. Thus, from the Banach closed graph theorem, there exists a constant C such that

$$\|\mathbf{u}\|_{DC} \le C \|\nabla \times \mathbf{u}\|_{H(\text{div})},$$

for every $\mathbf{u} \in \mathcal{L}$, which implies

$$\left(\|\mathbf{u}\|^2 + \|\nabla \times \mathbf{u}\|^2\right)^{1/2} \le C \|\nabla \times \mathbf{u}\|,$$

which, in turn, yields (2.20) with $C_1 = \sqrt{C^2 + 1}$.

To prove (2.21), let $\mathbf{u} \in DC_0/Q$. We seek a decomposition of $\mathbf{u}$ of the form

$$\mathbf{u} = \nabla p + \nabla \times \mathbf{v},$$

where $\nabla p \in DC_0/Q$. In particular, we construct p so that $\mathbf{u} - \nabla p \in \mathcal{L}$.

This is accomplished by first computing p_0 that satisfies

$$\begin{cases} \Delta p_0 = \nabla \cdot \mathbf{u} & \text{in} \quad \Omega, \\ p_0 = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.22}$$

Clearly, $\nabla p_0 \in DC_0$ and $\|\nabla p_0\| \le C \|\nabla \cdot \mathbf{u}\|$.

Next, define

$$\nabla p = \nabla p_0 - \sum_{j=1}^{m} \beta_j \nabla \zeta_j, \tag{2.23}$$

where ζ_j is defined in (2.18). The coefficients β_j, $j = 1, \ldots, m$, are chosen to satisfy

$$\sum_{j=1}^{m} \langle \mathbf{n} \cdot \nabla \zeta_j, 1 \rangle_{\Gamma_i} \beta_j = \langle \mathbf{n} \cdot \nabla p_0, 1 \rangle_{\Gamma_i} \tag{2.24}$$

for $i = 1, \ldots, m$. This $m \times m$ system is nonsingular. To see this, assume not. Then, there exist nonzero $\hat{\beta}_j$, $j = 1, \ldots, m$, satisfying (2.24) with zero right-hand side. This implies that

$$\phi := \sum_{j=1}^{m} \hat{\beta}_j \nabla \zeta_j \in \mathcal{L}.$$

However, $\mathcal{L} \perp Q$, which is a contradiction.

Now, the coefficients of the system (2.24) depend only on the domain Ω. Thus, there are generic constants C such that

$$\max_{j=1,\dots,m} |\beta_j| \le C \max_{j=1,\dots,m} |\langle \mathbf{n}\cdot\nabla p_0,\, 1\rangle_{\Gamma_i}| \le C\|\mathbf{n}\cdot\nabla p_0\|_{-1/2,\partial\Omega} \le C\|\nabla p_0\|_{H(\mathrm{div})} \le C\|\nabla\cdot\mathbf{u}\|,$$

which implies that there is a constant C such that

$$\|\nabla p\| \le C\|\nabla\cdot\mathbf{u}\|. \tag{2.25}$$

Having established that $\mathbf{u} - \nabla p \in \mathcal{L}$, from (2.20) we have

$$\|\mathbf{u} - \nabla p\| \le C_1\|\nabla\times\mathbf{u}\|.$$

Together with (2.25), this implies

$$\|\mathbf{u}\| \le \|\mathbf{u} - \nabla p\| + \|\nabla p\| \le C_2\left(\|\nabla\times\mathbf{u}\| + \|\nabla\cdot\mathbf{u}\|\right),$$

which completes the proof. ∎

We are now in a position to define the dual norms on D_0C, DC_0/Q, and $\mathcal{L}$. Let Ω be simply connected and define

$$\|\mathbf{u}\|_{-D_0C} := \sup_{\mathbf{w}\in D_0C} \frac{\langle \mathbf{u},\, \mathbf{w}\rangle}{|\mathbf{w}|_{DC}}, \tag{2.26}$$

$$\|\mathbf{u}\|_{-DC_0/Q} := \sup_{\mathbf{w}\in DC_0/Q} \frac{\langle \mathbf{u},\, \mathbf{w}\rangle}{|\mathbf{w}|_{DC}}, \tag{2.27}$$

$$\|\mathbf{u}\|_{-\mathcal{L}} := \sup_{\mathbf{w}\in\mathcal{L}} \frac{\langle \mathbf{u},\, \mathbf{w}\rangle}{|\mathbf{w}|_{DC}} = \sup_{\mathbf{w}\in\mathcal{L}} \frac{\langle \mathbf{u},\, \mathbf{w}\rangle}{\|\nabla\times\mathbf{w}\|}. \tag{2.28}$$

The following lemma provides some tools that will be used in the next section.

LEMMA 2.3. *Let C be a generic constant depending only on Ω. The following inequalities hold whenever the right-hand side is bounded:*

$$\|\mathbf{u}\|_{-\mathcal{L}} \le \|\mathbf{u}\|_{-DC_0/Q}, \tag{2.29}$$

$$\|\mathbf{u}\|_{-1,0} \le C\|\mathbf{u}\|_{-DC_0/Q}, \tag{2.30}$$

$$\|\mathbf{u}\|_{-1,0} \le C\|\mathbf{u}\|_{-D_0C}, \tag{2.31}$$

$$\|\nabla\times\mathbf{u}\|_{-DC_0/Q} = \|\nabla\times\mathbf{u}\|_{-\mathcal{L}}. \tag{2.32}$$

Proof. The first inequality follows from the fact that $\mathcal{L} \subset DC_0/Q$. For the other inequalities, we make use of the inclusions $(H_0^1(\Omega))^3 \subset DC_0/Q$ and $(H_0^1(\Omega))^3 \subset D_0C$ and the bound $|\mathbf{z}|_{DC} \le C|\mathbf{z}|_1$ for some constant C. We have

$$\begin{aligned}\|\mathbf{u}\|_{-1,0} &= \sup_{\mathbf{z}\in(H_0^1(\Omega))^3} \frac{\langle \mathbf{u},\, \mathbf{z}\rangle}{|\mathbf{z}|_1} \le C \sup_{\mathbf{z}\in(H_0^1(\Omega))^3} \frac{\langle \mathbf{u},\, \mathbf{z}\rangle}{|\mathbf{z}|_{DC}} \\ &\le C \sup_{\mathbf{z}\in DC_0/Q} \frac{\langle \mathbf{u},\, \mathbf{z}\rangle}{|\mathbf{z}|_{DC}} = C\|\mathbf{u}\|_{-DC_0/Q}.\end{aligned}$$

The third inequality follows in the same manner. For the final equality, we use the fact that, for any $\mathbf{z} \in DC_0/Q$, there is a $\hat{\mathbf{z}} \in \mathcal{L}$ such that $\nabla\times\mathbf{z} = \nabla\times\hat{\mathbf{z}}$. This fact is

established in a manner similar to the construction of $\mathbf{u} - \nabla p$ in the proof of Lemma 2.2. Thus,

$$\|\nabla\times\mathbf{u}\|_{-DC_0/Q} = \sup_{\mathbf{z}\in DC_0/Q} \frac{\langle \mathbf{u},\, \nabla\times\mathbf{z}\rangle}{|\mathbf{z}|_{DC}} \le \sup_{\mathbf{z}\in DC_0/Q} \frac{\langle \mathbf{u},\, \nabla\times\mathbf{z}\rangle}{\|\nabla\times\mathbf{z}\|}$$
$$\le \sup_{\hat{\mathbf{z}}\in\mathcal{L}} \frac{\langle \mathbf{u},\, \nabla\times\hat{\mathbf{z}}\rangle}{\|\nabla\times\hat{\mathbf{z}}\|} = \|\nabla\times\mathbf{u}\|_{-\mathcal{L}}.$$

Together with (2.29), this yields (2.32). ∎

We now say a few words about the evaluation of these dual norms. Since DC is compact in $L^2(\Omega)$, one can show that the supremum is achieved in each of the definitions above. This leads to the following result.

THEOREM 2.4. *The dual spaces defined above are Hilbert spaces with inner products defined as follows:*

$$\langle \mathbf{u},\, \mathbf{v}\rangle_{-D_0C} := \langle \mathbf{z},\, \mathbf{v}\rangle, \tag{2.33}$$

where $\mathbf{z} \in D_0C$ *is the unique solution of*

$$\begin{aligned} \langle \nabla\times\mathbf{z},\, \nabla\times\mathbf{w}\rangle + \langle \nabla\cdot\mathbf{z},\, \nabla\cdot\mathbf{w}\rangle &= \langle \mathbf{u},\, \mathbf{w}\rangle \qquad \forall\ \mathbf{w} \in D_0C; \\ \langle \mathbf{u},\, \mathbf{v}\rangle_{-DC_0/Q} &:= \langle \mathbf{z},\, \mathbf{v}\rangle, \end{aligned} \tag{2.34}$$

where $\mathbf{z} \in DC_0/Q$ *is the unique solution of*

$$\begin{aligned} \langle \nabla\times\mathbf{z},\, \nabla\times\mathbf{w}\rangle + \langle \nabla\cdot\mathbf{z},\, \nabla\cdot\mathbf{w}\rangle &= \langle \mathbf{u},\, \mathbf{w}\rangle \qquad \forall\ \mathbf{w} \in DC_0/Q; \\ \langle \mathbf{u},\, \mathbf{v}\rangle_{-\mathcal{L}} &:= \langle \mathbf{z},\, \mathbf{v}\rangle, \end{aligned} \tag{2.35}$$

where $\mathbf{z} \in \mathcal{L}$ *is the unique solution of*

$$\langle \nabla\times\mathbf{z},\, \nabla\times\mathbf{w}\rangle = \langle \mathbf{u},\, \mathbf{w}\rangle \qquad \forall\ \mathbf{w} \in \mathcal{L}.$$

Proof. We offer only a sketch of the proof. Standard compactness arguments yield the existence of the supremum $\mathbf{z}$ in each case. The weak forms follow from a perturbation argument. ∎

Section 3.3 will contain some remarks about the numerical evaluation of these norms.

3. FOSLS. The primary objective of this section is to establish product norm equivalence of the homogeneous part of the least-squares functionals based on (2.7) and (2.8) in appropriate Sobolev spaces. To this end, assume that $\mathbf{f} \in L^2(\Omega)^3 \cap \mathcal{N}^\perp$ and define the following:

$$G(\mathbf{U},\, \mathbf{u};\, \mathbf{f}) = \|\mathbf{f} + \nabla\cdot A\mathbf{U}\|^2 + \|\nabla\times\mathbf{U}\|^2 + \langle A(\mathbf{U} - \nabla\mathbf{u}), (\mathbf{U} - \nabla\mathbf{u})\rangle \tag{3.1}$$

for $(\mathbf{U},\, \mathbf{u}) \in \mathcal{V} \times \mathcal{Y}$;

$$G_{-1}(\mathbf{U};\, \mathbf{f}) = \|\mathbf{f} + \nabla\cdot A\mathbf{U}\|^2_{-1} + \|\nabla\times\mathbf{U}\|^2_{-\mathcal{L}} \tag{3.2}$$

for $\mathbf{U} \in \mathcal{U}$; and

$$G_0(\mathbf{U};\, \mathbf{f}) = \|\mathbf{f} + \nabla\cdot A\mathbf{U}\|^2 + \|\nabla\times\mathbf{U}\|^2 \tag{3.3}$$

for $\mathbf{U} \in \mathcal{V}$.

In what follows, C, possibly with subscripts, will denote a generic constant that may vary in meaning with each occurrence and may depend on Ω but is independent of the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$. We will frequently use the term *uniform* in reference to a relation to mean that it is independent of ν.

We first establish uniform continuity and coercivity (i.e., equivalence) of the respective homogeneous functionals $G_{-1}(\mathbf{U};\, \mathbf{0})$ and $G_0(\mathbf{U};\, \mathbf{0})$ in terms of the squared product norms $M_{-1}(\mathbf{U})$ and $M_0(\mathbf{U})$ defined on the spaces $\mathcal{U}$ and $\mathcal{V}$ by

$$M_{-1}(\mathbf{U}) = \|\mathbf{U}\|^2 + \lambda^2 \|\mathrm{tr}\mathbf{U}\|^2$$

and

$$M_0(\mathbf{U}) = \|\mathbf{U}\|_1^2 + \lambda^2 \|\nabla \mathrm{tr}\mathbf{U}\|^2.$$

We then show that $G(\mathbf{U},\, \mathbf{u};\, \mathbf{0})$ is uniformly equivalent to the modified product H^1-type norm defined on $\mathcal{V} \times \mathcal{Y}$ by

$$M(\mathbf{U},\, \mathbf{u}) = \|\mathbf{U}\|_1^2 + \lambda^2 \|\nabla \mathrm{tr}\mathbf{U}\|^2 + |\lambda|\, \|\mathrm{tr}\mathbf{U}\|^2 + \|A^{\frac{1}{2}} \nabla \mathbf{u}\|^2.$$

In what follows, we appeal to standard H^1 regularity estimates (cf. [12, 13]) for elasticity equation (2.5):

$$\|\mathbf{v}\|_1 + \lambda \|\nabla \cdot \mathbf{v}\| \leq C\, \|\nabla \cdot A \nabla \mathbf{v}\|_{-1} \tag{3.4}$$

for any $\mathbf{v} \in \mathcal{W}$. If the domain Ω has $C^{1,\,1}$ boundary, then we may appeal to standard H^2 regularity results (cf. [12, 13]):

$$\|\mathbf{v}\|_2 \leq C\, \|\nabla \cdot A \nabla \mathbf{v}\| \tag{3.5}$$

for any $\mathbf{v} \in \mathcal{W} \cap H^2(\Omega)^3$. (If $\nabla \cdot A \nabla \mathbf{v} \in L^2(\Omega)^3$, then H^2 regularity implies that $\mathbf{v} \in H^2(\Omega)^3$.)

The equivalence of G_{-1} and M_{-1} is established directly in section 3.1, while the equivalence of G_0 and M_0 is established by using a regularity result for an auxiliary Stokes-like system, which we describe in section 3.2. In section 3.2, we prove the equivalence for G_0 using this regularity result. Finally, the regularity result for the Stokes-like system is established in section 4.

Toward this end, define the matrix

$$\mathcal{P} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.6}$$

The operator $\mathcal{P}$ performs the transpose when the vectors are viewed as 3×3 matrices;

that is, $(\mathcal{P}\mathbf{U})_{3\times 3} = (\mathbf{U}_{3\times 3})^t$. Next, define

$$\mathcal{Q} := \mathcal{P} - \frac{1}{2}\mathbf{b}\mathbf{b}^t = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \tag{3.7}$$

where $\mathbf{b}$ is defined in (2.4). It is easy to see that $\|\mathcal{Q}\| = 1$ and $\mathcal{Q}^{-1} = \mathcal{P} - \mathbf{b}\mathbf{b}^t$ with $\|\mathcal{Q}^{-1}\| = 2$. Let I_9 be the 9×9 identity matrix and define the constants

$$c := \frac{1}{2\mu} \quad \text{and} \quad d := \frac{\lambda}{2\mu(3\lambda + 2\mu)} \tag{3.8}$$

and the matrices

$$\mathcal{M} := cI_9 - d\mathbf{b}\mathbf{b}^t, \tag{3.9}$$

$$\mathcal{R} := \left[(c-d)I_9 + d\mathcal{P}\right]. \tag{3.10}$$

The matrix $\mathcal{M}$ will be important in the proof of equivalence for G_{-1}, while the matrix $\mathcal{R}$ will play a central role in the proof for G_0.

Note that if $\mathbf{N} \in \text{Null}(A)$, then it has the form

$$\mathbf{N} = (0,\, N_{12},\, -N_{13},\; -N_{12},\, 0,\, N_{23},\; N_{13},\, -N_{23},\, 0)^t.$$

Define $\boldsymbol{\eta} = (N_{23},\, N_{13},\, N_{12})^t$. Then, one can show that

$$\langle \mathbf{N},\, \nabla\times\,(\mathcal{Q}\mathbf{v})\rangle = \langle \boldsymbol{\eta},\, \nabla\cdot\mathbf{v}\rangle \tag{3.11}$$

for $\mathbf{v} \in (H^1(\Omega))^9$, and

$$\|\mathbf{N}\| = \sqrt{2}\|\boldsymbol{\eta}\|. \tag{3.12}$$

To formalize the relationship between $\mathbf{N}$ and $\boldsymbol{\eta}$, define $\mathcal{J}_{9\times 3}$ to be the matrix such that

$$\mathbf{N} = \mathcal{J}\boldsymbol{\eta}. \tag{3.13}$$

Note that the columns of $\mathcal{J}$ are a basis for the null space of A_2.

3.1. Equivalence of G_{-1} and M_{-1}. We first prove the following useful result.

LEMMA 3.1. *Let $p \in L^2(\Omega)$. Then*

$$\|p\| \le \sup_{\mathbf{v}\in H^1(\Omega)^3} \frac{\langle p, \nabla\cdot\mathbf{v}\rangle}{|\mathbf{v}|_1}.$$

Proof. In this proof, we use subscripts to denote the domain of norms. First, extend Ω to some bounded convex domain, $\hat{\Omega}$, with C^2 boundary and extend p to $\hat{\Omega}$

by setting $p = 0$ in $\hat{\Omega}/\Omega$. Since we know that there exists q in $H^2(\hat{\Omega})$ with $\Delta q = p$, and thus $\mathbf{v} = \nabla q \in H^1(\hat{\Omega})^3$ with $\nabla \cdot \mathbf{v} = p$ on $\hat{\Omega}$, we have

$$\begin{aligned}\|p\|_{\Omega} = \|p\|_{\hat{\Omega}} &= \sup_{\mathbf{v}\in H^1(\hat{\Omega})^3} \frac{\langle p, \nabla\cdot\mathbf{v}\rangle_{\hat{\Omega}}}{\|\nabla\cdot\mathbf{v}\|_{\hat{\Omega}}} \\ &= \sup_{\mathbf{v}\in H^1(\hat{\Omega})^3} \frac{\langle p, \nabla\cdot\mathbf{v}\rangle_{\hat{\Omega}}}{\left(\|\nabla\cdot\mathbf{v}\|^2_{\hat{\Omega}} + \|\nabla\times\mathbf{v}\|^2_{\hat{\Omega}}\right)^{1/2}}.\end{aligned}$$

Now, using [11, Chapter I, section 3.5, Lemma 3.8] and the fact that $\hat{\Omega}$ is convex, we have

$$\|p\|_{\Omega} \le \sup_{\mathbf{v}\in H^1(\hat{\Omega})^3} \frac{\langle p, \nabla\cdot\mathbf{v}\rangle_{\hat{\Omega}}}{|v|_{1,\hat{\Omega}}} \le \sup_{\mathbf{v}\in H^1(\Omega)^3} \frac{\langle p, \nabla\cdot\mathbf{v}\rangle_{\Omega}}{|v|_{1,\Omega}},$$

which yields the result. □

We use Lemma 3.1 to establish the following lemma.

LEMMA 3.2. *Let* $\mathbf{Z} \in L^2(\Omega)^9$ *such that* $\langle \mathbf{Z}, \nabla\mathbf{v}\rangle = 0 \ \forall\ \mathbf{v} \in H^1(\Omega)^3$. *Then*

$$\|\mathrm{tr}(\mathbf{Z})\,\|^2 \le \frac{9}{4}\|\mathbf{Z}\|^2.$$

Proof. Consider such a $\mathbf{Z}$ and its pointwise orthogonal decomposition

$$\mathbf{Z} = \frac{\alpha}{3}\mathbf{b} + \mathbf{C},$$

where $\mathbf{b}$ is defined in (2.4) and $\mathrm{tr}(\mathbf{C}) = 0$. Thus, $\alpha = \mathrm{tr}(\mathbf{Z})$ and

$$\|\mathbf{Z}\|^2 = \frac{1}{3}\|\alpha\|^2 + \|\mathbf{C}\|^2. \tag{3.14}$$

For $\mathbf{v} \in H^1(\Omega)^3$, we have

$$\begin{aligned}0 &= \langle \mathbf{Z}, \nabla\mathbf{v}\rangle \\ &= \frac{1}{3}\langle \alpha\mathbf{b}, \nabla\mathbf{v}\rangle + \langle \mathbf{C}, \nabla\mathbf{v}\rangle \\ &= \frac{1}{3}\langle \alpha, \nabla\cdot\mathbf{v}\rangle + \langle \mathbf{C}, \nabla\mathbf{v}\rangle.\end{aligned}$$

Thus, by Lemma 3.1, we have

$$\|\alpha\| \le \sup_{\mathbf{v}\in H^1(\Omega)^3} \frac{\langle \alpha, \nabla\cdot\mathbf{v}\rangle}{|\mathbf{v}|_1} = 3 \sup_{\mathbf{v}\in H^1(\Omega)^3} \frac{\langle \mathbf{C}, \nabla\mathbf{v}\rangle}{|\mathbf{v}|_1} = 3\|\mathbf{C}\|. \tag{3.15}$$

Combining (3.15) and (3.14) yields the result. □

THEOREM 3.1. *Assume that* Ω *is simply connected. Then, the functionals* $G_{-1}(\mathbf{U};\,\mathbf{0})$ *and* $M_{-1}(\mathbf{U})$ *satisfy the uniform equivalence relation*

$$\frac{1}{C}\,M_{-1}(\mathbf{U}) \le G_{-1}(\mathbf{U};\,\mathbf{0}) \le C M_{-1}(\mathbf{U}) \tag{3.16}$$

$\forall\ \mathbf{U} \in \mathcal{U}$.

Proof. The upper bound in (3.16) follows from the easily established bounds

$$\|\nabla \cdot A\mathbf{U}\|_{-1} \leq \|A\mathbf{U}\| \quad \text{and} \quad \|\nabla \times \mathbf{U}\|_{-\mathcal{L}} \leq \|\mathbf{U}\|,$$

from the triangle inequality, and from noting that $A_1\mathbf{U} = (\mathrm{tr}\mathbf{U})\mathbf{b}$. To show the validity of the lower bound in (3.16), we first show that

$$\frac{1}{C}\|\mathbf{U}\|^2 \leq G_{-1}(\mathbf{U}; \mathbf{0}) \qquad \forall\ \mathbf{U} \in \mathcal{U}. \tag{3.17}$$

To this end, write $\mathbf{U} \in \mathcal{U}$ as follows:

$$\mathbf{U} = \nabla\mathbf{v} + \mathbf{V} + \mathbf{N}, \tag{3.18}$$

where $\mathbf{v} \in \mathcal{W}$ solves (2.5) with $\mathbf{f} = -\nabla \cdot A\mathbf{U}$ and the decomposition $\mathbf{V} + \mathbf{N} = \mathbf{U} - \nabla\mathbf{v}$ is characterized by restricting $\mathbf{V}$ to the range of A_2 and $\mathbf{N}$ to the null space of A_2. Note that $\mathbf{V}$ is symmetric and that $\mathbf{N} = (0,\ p,\ -q,\ p,\ 0,\ r,\ q,\ -r,\ 0)^t$ for some $p,\ q,\ r \in L_0^2(\Omega)$ ($\int_\Omega p\,dx = \int_\Omega q\,dx = \int_\Omega r\,dx = 0$ because $\mathbf{U} \in (\nabla\mathcal{N})^\perp$). Let $\boldsymbol{\eta} = (r,\ p,\ q)^t$, which implies $\mathbf{N} = \mathcal{J}\boldsymbol{\eta}$. Hence, we have that

$$A_2\mathbf{V} = \mathbf{V} \tag{3.19}$$

and, by (3.11), that

$$\langle \mathbf{N},\ \nabla\times(\mathcal{Q}\mathbf{v})\rangle = \langle \boldsymbol{\eta},\ \nabla \cdot \mathbf{v}\rangle \tag{3.20}$$

for $\mathbf{v} \in (H^1)^9$. Now, the definitions of A, A_1, A_2, and $\mathbf{N}$ imply that $A\mathbf{N} = \lambda A_1\mathbf{N} + 2\mu A_2\mathbf{N} = \mathbf{0}$. Thus, since

$$\nabla \cdot A\mathbf{V} = \nabla \cdot A(\mathbf{V} + \mathbf{N}) = \nabla \cdot A\mathbf{U} - \nabla \cdot A\nabla\mathbf{v} = \mathbf{0} \quad \text{in} \quad \Omega$$

and

$$\mathbf{n} \cdot\ A\mathbf{V} = \mathbf{0} \quad \text{on} \quad \partial\Omega, \tag{3.21}$$

it then follows (cf. [11, Chapter I, section 3.3, Theorem 3.6]) that

$$\begin{aligned} A\mathbf{V} &= \nabla\times\mathbf{W} && \text{in} && \Omega, \\ \nabla \cdot \mathbf{W} &= \mathbf{0} && \text{in} && \Omega, \\ \mathbf{n} \times \mathbf{W} &= \mathbf{0} && \text{on} && \partial\Omega, \\ \langle \mathbf{n} \cdot \mathbf{W},\ 1\rangle_{\Gamma_i} &= 0 && \text{for} && i = 1, \ldots, m, \end{aligned} \tag{3.22}$$

for some $\mathbf{W} \in \mathcal{L}^3$. By taking the trace of both sides of (3.22) and using (3.19), we see that

$$\mathrm{tr}\mathbf{V} = \frac{1}{3\lambda + 2\mu}\mathrm{tr}\left(\nabla \times \mathbf{W}\right), \tag{3.23}$$

which, using (3.19) again, implies that

$$2\mu\mathbf{V} = 2\mu A_2\mathbf{V} = A\mathbf{V} - \lambda A_1\mathbf{V} = \nabla\times\mathbf{W} - \lambda(\mathrm{tr}\mathbf{V})\,\mathbf{b}$$

or

$$\mathbf{V} = \frac{1}{2\mu}\left(\nabla\times\mathbf{W} - \frac{\lambda}{3\lambda + 2\mu}\left(\mathrm{tr}(\nabla \times \mathbf{W})\,\mathbf{b}\right)\right) = \mathcal{M}\nabla\times\mathbf{W}, \tag{3.24}$$

where $\mathcal{M}$ is defined in (3.9).

Since $\nabla\times\mathbf{W}$ is symmetric and $\mathbf{N}$ is skew-symmetric, (3.21), (3.22), and (3.24) yield

$$\begin{aligned}\langle \mathbf{U}, \nabla\times\mathbf{W}\rangle &= \langle \nabla\mathbf{v}, \nabla\times\mathbf{W}\rangle + \langle \mathbf{V}, \nabla\times\mathbf{W}\rangle + \langle \mathbf{N}, \nabla\times\mathbf{W}\rangle \\ &= \langle \mathbf{V}, \nabla\times\mathbf{W}\rangle \\ &= \langle \mathcal{M}\nabla\times\mathbf{W}, \nabla\times\mathbf{W}\rangle \\ &= c\|\nabla\times\mathbf{W}\|^2 - d\|\mathrm{tr}(\nabla\times\mathbf{W})\|^2.\end{aligned}$$

By Lemma 3.2, we have

$$\langle \mathbf{U}, \nabla\times\mathbf{W}\rangle \geq \left(c - \frac{9}{4}d\right)\|\nabla\times\mathbf{W}\|^2 \geq \frac{1}{8\mu}\|\nabla\times\mathbf{W}\|^2.$$

Thus, from (2.28),

$$\frac{1}{8\mu}\|\nabla\times\mathbf{W}\| \leq \frac{\langle \mathbf{U}, \nabla\times\mathbf{W}\rangle}{\|\nabla\times\mathbf{W}\|} \leq \|\nabla\times\mathbf{U}\|_{-\mathcal{L}}. \tag{3.25}$$

Now

$$\|\mathbf{V}\| = \|\mathcal{M}\nabla\times\mathbf{W}\| \leq \|\mathcal{M}\|\,\|\nabla\times\mathbf{W}\| = \frac{1}{2\mu}\|\nabla\times\mathbf{W}\|,$$

which yields

$$\|\mathbf{V}\| \leq 4\|\nabla\times\mathbf{U}\|_{-\mathcal{L}}. \tag{3.26}$$

Using [11, Corollary 2.4], (3.18), (3.20), (3.4), (2.30), and (2.32), we have

$$\begin{aligned}\frac{1}{\sqrt{2}}\|\mathbf{N}\| = \|\boldsymbol{\eta}\| &\leq C \sup_{\mathcal{Z}\in H_0^1(\Omega)^9} \frac{\langle \boldsymbol{\eta}, \nabla\cdot\mathcal{Z}\rangle}{|\mathcal{Z}|_1} \\ &= C \sup_{\mathcal{Z}\in H_0^1(\Omega)^9} \frac{\langle \mathbf{N}, \nabla\times\mathcal{Q}\mathcal{Z}\rangle}{|\mathcal{Z}|_1} \\ &= C \sup_{\mathcal{Z}\in H_0^1(\Omega)^9} \frac{\langle \mathbf{N}, \nabla\times\mathcal{Z}\rangle}{|\mathcal{Z}|_1} \\ &= C \sup_{\mathcal{Z}\in H_0^1(\Omega)^9} \left(\frac{\langle \mathbf{U}, \nabla\times\mathcal{Z}\rangle}{|\mathcal{Z}|_1} - \frac{\langle \nabla\mathbf{v}, \nabla\times\mathcal{Z}\rangle}{|\mathcal{Z}|_1} - \frac{\langle \mathbf{V}, \nabla\times\mathcal{Z}\rangle}{|\mathcal{Z}|_1}\right) \\ &\leq C\left(\|\nabla\times\mathbf{U}\|_{-1,0} + \|\nabla\mathbf{v}\| + \|\mathbf{V}\|\right) \\ &\leq C\left(\|\nabla\times\mathbf{U}\|_{-\mathcal{L}} + \|\nabla\cdot A\mathbf{U}\|_{-1}\right).\end{aligned} \tag{3.27}$$

Combining (3.26) and (3.27) with (3.4), we have

$$\begin{aligned}G_{-1}(\mathbf{U}, \mathbf{0}) &= \|\nabla\times\mathbf{U}\|_{-\mathcal{L}}^2 + \|\nabla\cdot A\mathbf{U}\|_{-1}^2 \\ &\geq \frac{1}{C}\left(\|\nabla\mathbf{v}\|^2 + \|\mathbf{V}\|^2 + \|\boldsymbol{\eta}\|^2\right) \\ &\geq \frac{1}{C}\|\mathbf{U}\|^2.\end{aligned}$$

For the second term in M_{-1}, using (3.18) and (3.23) (but now stemming from $\mathbf{U} \in \mathcal{U}$), we note that

$$\begin{aligned}\lambda A_1 \mathbf{U} &= \lambda A_1 \nabla \mathbf{v} + \lambda A_1 \mathbf{V} \\ &= \left(\lambda \nabla \cdot \mathbf{v} + \frac{\lambda}{3\lambda + 2\mu} \mathrm{tr}\left(\nabla \times \mathbf{W}\right) \right) \mathbf{b}.\end{aligned}$$

Hence,

$$\lambda \mathrm{tr}\mathbf{U} = \lambda \nabla \cdot \mathbf{v} + \frac{\lambda}{3\lambda + 2\mu} \mathrm{tr}\left(\nabla \times \mathbf{W}\right).$$

It then follows from (3.4) and (3.25) that

$$\begin{aligned}\lambda \|\mathrm{tr}\mathbf{U}\| &\le \|\lambda \nabla \cdot \mathbf{v}\| + \|\nabla \times \mathbf{W}\| \\ &= C \left(\|\nabla \cdot A\mathbf{U}\|_{-1} + \|\nabla \times \mathbf{U}\|_{-\mathcal{L}} \right). \qquad \square\end{aligned}$$

REMARK 3.1. *The norm, $\|\cdot\|_{-\mathcal{L}}$, is the smallest norm for which this line of proof works. Section* 3.3 *will reveal some numerical advantages to this seemingly strange choice. From* (2.29) *and* (2.32), *we see that one could also use $\|\cdot\|_{-DC_0/Q}$ in the definition of G_{-1} in* (3.2). *When Ω is a convex polyhedron or $\partial\Omega \in C^{1,1}$, then DC_0 is continuously embedded in $H^1(\Omega)$ (see* [11, *Chapter* I, *section* 3.4, *Theorem* 3.7]). *In this case, one may use the norm*

$$\|\mathbf{u}\|_{-1,DC_0} := \sup_{\mathbf{w} \in DC_0} \frac{\langle \mathbf{u}, \mathbf{w} \rangle}{|\mathbf{w}|_1} \tag{3.28}$$

in (3.2). *If we use $\|\mathbf{w}\|_1$ in* (3.28), *we may also remove the restriction that Ω be simply connected. The proofs of these extensions are omitted because they are long but straightforward.*

3.2. Equivalence of G_0 and M_0. As in the proof of Theorem 3.1, we need to decompose $\mathbf{U}$ as in (3.18) and $A\mathbf{V}$ as in (3.22). Now, however, we choose $\mathbf{W}$ to satisfy $\mathbf{n} \cdot \mathbf{W} = \mathbf{0}$ on $\partial\Omega$. Recall from (3.24) that

$$\mathbf{V} = \mathcal{M} \nabla \times \mathbf{W} = c \nabla \times \mathbf{W} - d\, \mathrm{tr}(\nabla \times \mathbf{W}) \mathbf{b}.$$

We now show that

$$\nabla \times \mathbf{V} = -\mathcal{R} \Delta \mathbf{W},$$

where $\mathcal{R}$ is defined in (3.10).

LEMMA 3.3. *If $\mathbf{U} = \nabla \times \mathbf{W} \in \mathrm{Range}(A)$ and $\nabla \cdot \mathbf{W} = \mathbf{0}$, then*

$$\begin{aligned}\nabla \times (\mathrm{tr}(\nabla \times \mathbf{W}) \mathbf{b}) = (0,\, &\Delta w_{21} - \Delta w_{12},\, \Delta w_{31} - \Delta w_{13},\, \Delta w_{12} - \Delta w_{21}, \\ &0,\, \Delta w_{32} - \Delta w_{23},\, \Delta w_{13} - \Delta w_{31},\, \Delta w_{23} - \Delta w_{32},\, 0)^t.\end{aligned} \tag{3.29}$$

Proof. Since $\nabla \times \mathbf{W} \in \mathrm{Range}(A)$, then $\nabla \times \mathbf{W}$ is symmetric, which with $\nabla \cdot \mathbf{W} = \mathbf{0}$ and a somewhat lengthy calculation yields (3.29). $\square$

But (3.22) implies that $\nabla \times \nabla \times \mathbf{W} = -\Delta \mathbf{W}$, which, together with Lemma 3.3 and (3.24), implies that

$$\begin{aligned}\nabla \times \mathbf{V} &= -c\Delta \mathbf{W} - d \nabla \times \left(\mathrm{tr}(\nabla \times \mathbf{W})\, \mathbf{b} \right) \\ &= (-c\Delta w_{11},\, (d-c)\Delta w_{12} - d\Delta w_{21},\, (d-c)\Delta w_{13} - d\Delta w_{31}, \\ &\quad (d-c)\Delta w_{21} - d\Delta w_{12},\, -c\Delta w_{22},\, (d-c)\Delta w_{23} - d\Delta w_{32}, \\ &\quad (d-c)\Delta w_{31} - d\Delta w_{13},\, (d-c)\Delta w_{32} - d\Delta w_{23},\, -c\Delta w_{33})^t,\end{aligned} \tag{3.30}$$

where c and d are defined in (3.8).

One can now write

$$\nabla\times\mathbf{V} = -\mathcal{R}\Delta\mathbf{W}. \tag{3.31}$$

From the decomposition of $\mathbf{U}$ and (3.13) we have

$$\begin{aligned}\nabla\times\mathbf{U} &= \nabla\times\mathbf{V} + \nabla\times\mathbf{N}\\ &= -\Delta\mathcal{R}\mathbf{W} + \nabla\times\mathcal{J}\boldsymbol{\eta}.\end{aligned} \tag{3.32}$$

Collecting all the properties of $\mathbf{W}$ and $\boldsymbol{\eta}$ leads to the Stokes-like system (3.33). To obtain the H^2 regularity that we need, we included a curl-symmetry condition in Ω, but we also require extra boundary conditions to obtain uniqueness. In general, if a system of equations is posed on a space with sufficient smoothness, then the equations themselves can be imposed on the boundary whenever there is reason to do so. We found it necessary to impose the divergence and curl-symmetry equations on $\partial\Omega$. We have enough smoothness to do this because the vector variable components are assumed to be in $H^2(\Omega)$.

$$\left\{\begin{array}{rcll} -\mathcal{R}\Delta\mathbf{W} + \nabla\times\mathcal{J}\boldsymbol{\eta} &=& \mathbf{F} & \text{in } \Omega,\\ \nabla\cdot\mathbf{W} &=& \mathbf{0} & \text{in } \Omega,\\ \nabla\times\mathbf{W} - (\nabla\times\mathbf{W})^t &=& \mathbf{0} & \text{in } \Omega,\\ \mathbf{n}\cdot\mathbf{W} &=& \mathbf{0} & \text{on } \partial\Omega,\\ \mathbf{n}\cdot\nabla\times\mathbf{W} &=& \mathbf{0} & \text{on } \partial\Omega,\\ \nabla\cdot\mathbf{W} &=& \mathbf{0} & \text{on } \partial\Omega,\\ \nabla\times\mathbf{W} - (\nabla\times\mathbf{W})^t &=& \mathbf{0} & \text{on } \partial\Omega, \end{array}\right. \tag{3.33}$$

where $\mathbf{W}\in H^2(\Omega)^9$, $\boldsymbol{\eta}\in H^1(\Omega)^3\cap L^2_0(\Omega)^3$, and $\mathbf{F}\in L^2(\Omega)^9$.

In section 4, we establish the H^2 regularity estimate

$$\|\mathbf{W}\|_2^2 + \|\boldsymbol{\eta}\|_1^2 \le C\,\|\,-\mathcal{R}\Delta\mathbf{W} + \nabla\times\mathcal{J}\boldsymbol{\eta}\,\|^2. \tag{3.34}$$

We now prove Theorem 3.2 under assumptions (3.5) and (3.34).

THEOREM 3.2. *Suppose that Ω has $C^{1,1}$ boundary. Then the functionals $G_0(\mathbf{U};\mathbf{0})$ and $M_0(\mathbf{U})$ satisfy the uniform equivalence relation*

$$\frac{1}{C}M_0(\mathbf{U}) \le G_0(\mathbf{U};\mathbf{0}) \le CM_0(\mathbf{U}) \tag{3.35}$$

$\forall\ \mathbf{U}\in\mathcal{V}$.

Proof. Since the proof is similar to the previous theorem, we abbreviate the discussion here. The upper bound in (3.35) for G_0 is a straightforward consequence of the triangle inequality and the fact that

$$\nabla\cdot A_1\mathbf{U} = \nabla\mathrm{tr}\mathbf{U}. \tag{3.36}$$

The lower bound in (3.35) can be shown in the following way. From (3.32), we have that

$$\nabla\times\mathbf{U} = -\mathcal{R}\Delta\mathbf{W} + \nabla\times\mathbf{N}. \tag{3.37}$$

From (3.24), we have that

$$\|\mathbf{V}\|_1 \le C\left\|\frac{1}{2\mu}\mathbf{W}\right\|_2.$$

Combining (3.37) with (3.5), (3.34), and (3.18), we have that

$$\begin{aligned} G_0(\mathbf{U};\mathbf{0}) &= \|\nabla\cdot A\mathbf{U}\|^2 + \|\nabla\times\mathbf{U}\|^2 \\ &= \|\nabla\cdot A\nabla\mathbf{v}\|^2 + \|\nabla\times(\mathbf{V}+\mathbf{N})\|^2 \\ &= \|\nabla\cdot A\nabla\mathbf{v}\|^2 + \|-\mathcal{R}\Delta\mathbf{W} + \nabla\times\mathbf{N}\|^2 \\ &\geq \frac{1}{C}\left(\|\mathbf{v}\|_2^2 + \|\mathbf{W}\|_2^2 + \|\mathbf{N}\|_1^2\right) \\ &\geq \frac{1}{C}\left(\|\nabla\mathbf{v}\|_1^2 + \|\mathbf{V}\|_1^2 + \|\mathbf{N}\|_1^2\right) \\ &\geq \frac{1}{C}\|\mathbf{U}\|_1^2. \end{aligned} \tag{3.38}$$

For the second term, first note that the definition of A_2 implies that

$$\|2\mu\nabla\cdot A_2\mathbf{U}\|^2 \leq C\|\mathbf{U}\|_1^2. \tag{3.39}$$

From (3.36), the definition of A, the triangle inequality, (3.39), and (3.38), we have that

$$\begin{aligned} \|\lambda\nabla\mathrm{tr}\mathbf{U}\|^2 &\leq C\left(\|\nabla\cdot A\mathbf{U}\|^2 + \|2\mu\nabla\cdot A_2\mathbf{U}\|^2\right) \\ &\leq \left(\|\nabla\cdot A\mathbf{U}\|^2 + \|\mathbf{U}\|_1^2\right) \\ &\leq CG_0(\mathbf{U};0). \end{aligned}$$

This proves the bound (3.35). □

COROLLARY 3.1. *Assume that* (3.5) *and* (3.34) *hold. Then the functionals* $G(\mathbf{U},\mathbf{u};\mathbf{0})$ *and* $M(\mathbf{U},\mathbf{u})$ *satisfy the uniform equivalence relation*

$$\frac{1}{C}M(\mathbf{U},\mathbf{u}) \leq G(\mathbf{U},\mathbf{u};\mathbf{0}) \leq CM(\mathbf{U},\mathbf{u})$$

$\forall\ (\mathbf{U},\mathbf{u}) \in \mathcal{V}\times\mathcal{Y}$.

Proof. The proof of this corollary is straightforward and similar to that of the corollary in [8]. □

REMARK 3.2. *As in* [8], *we can develop a two-stage algorithm, where the first stage involves minimizing either* G_{-1} *or* G_0 *to compute* $\mathbf{U}$ *and the second stage appeals to the definition* $\mathbf{U} = \nabla\mathbf{u}$ *to recover* $\mathbf{u}$ *whenever it is needed.*

3.3. Implementation details. As in our previous work (cf. [6, 7, 8]), we consider replacing the minimization over the entire Hilbert space with minimization over a finite-dimensional subspace. In the case that $\partial\Omega$ has smooth boundary, that is, $\partial\Omega \in C^{1,1}$, and the functional G_0 is employed, standard C^0 finite element spaces could be used. Application of Ceá's lemma and approximation bounds for C^0 elements would then yield

$$\|\mathbf{U} - \mathbf{U}^h\| \leq Ch^\beta\|\mathbf{U}\|_{1+\beta} \tag{3.40}$$

for $\beta > 0$.

In the case that Ω does not have a $C^{1,1}$ boundary, there are several alternatives. The first is to approximate $\partial\Omega$ with a $C^{1,1}$ boundary, $\partial\hat{\Omega}$, and to use a finite element approximation as above.

A second alternative is to use a weighted L^2 functional. In the presence of reentrant corners and edges, the displacement behaves locally like r^α, where r is the

distance to the reentrant corner and $0 < \alpha < 1$. Thus, the displacement flux, $\mathbf{U}$, and hence the stress behave like $r^{\alpha-1}$ near the reentrant corner. Weights of the form $w(r) = r^\gamma$ should be chosen so that $\nabla\mathbf{U}$ is bounded in the weighted L^2 norm; that is, $\gamma > 2(1-\alpha)$. For details on the application of weighted L^2 norms in the context of FOSLS formulation for second-order elliptic equations, see [14].

A third approach is to use the G_{-1} functional. In this case, we use a finite element basis that is not necessarily C^0. This can be accomplished by considering the weak form associated with G_{-1}: find $\mathbf{U} \in \mathcal{U}$ such that

$$\langle \nabla \cdot A\mathbf{U}, \nabla \cdot A\mathbf{V}\rangle_{-1} + \langle \nabla\times\mathbf{U}, \nabla\times\mathbf{V}\rangle_{-\mathcal{L}} = \langle \mathbf{f}, \nabla \cdot A\mathbf{V}\rangle_{-1} \tag{3.41}$$

$\forall\ \mathbf{V} \in \mathcal{U}$. Similar to the results of Theorem 2.4, we have

$$\langle \nabla \cdot A\mathbf{U}, \nabla \cdot A\mathbf{V}\rangle_{-1} = \langle \nabla \cdot A\mathbf{U}, \mathbf{z}\rangle = \langle A\mathbf{U}, \nabla\mathbf{z}\rangle, \tag{3.42}$$

where $\mathbf{z} \in (H^1/\Re)^3$ satisfies

$$\langle \nabla\mathbf{z}, \nabla\mathbf{w}\rangle = \langle \nabla \cdot A\mathbf{V}, \mathbf{w}\rangle = \langle A\mathbf{V}, \nabla\mathbf{w}\rangle, \tag{3.43}$$

for every $\mathbf{w} \in (H^1/\Re)^3$. Equation (3.43) can be approximately solved using multilevel techniques (cf. [3]).

To approximate the right-hand side in (3.41), notice that it is a simple chore to find $\mathbf{Y} \in (H(\mathrm{div}))^3$ such that $\nabla \cdot \mathbf{Y} = \mathbf{f}$. Thus,

$$\langle \mathbf{f}, \nabla \cdot A\mathbf{V}\rangle_{-1} = \langle \mathbf{Y}, \nabla\mathbf{z}\rangle, \tag{3.44}$$

where $\mathbf{z}$ satisfies (3.43).

Similarly, the second term in (3.41) satisfies

$$\langle \nabla\times\mathbf{U}, \nabla\times\mathbf{V}\rangle_{-\mathcal{L}} = \langle \nabla\times\mathbf{U}, \mathbf{Z}\rangle = \langle \mathbf{U}, \nabla\times\mathbf{Z}\rangle, \tag{3.45}$$

where $\mathbf{Z} \in \mathcal{L}^3$ satisfies

$$\langle \nabla\times\mathbf{Z}, \nabla\times\mathbf{W}\rangle = \langle \nabla\times\mathbf{V}, \mathbf{W}\rangle = \langle \mathbf{V}, \nabla\times\mathbf{W}\rangle \tag{3.46}$$

$\forall\ \mathbf{W} \in \mathcal{L}^3$. Since $\nabla\times$ is a bijection from $\mathcal{L}$ to D_f, we can restate this as

$$\langle \nabla\times\mathbf{U}, \nabla\times\mathbf{V}\rangle_{-\mathcal{L}} = \langle \mathbf{U}, \mathbf{X}\rangle, \tag{3.47}$$

where $\mathbf{X} \in D_f^3$ satisfies

$$\langle \mathbf{X}, \mathbf{W}\rangle = \langle \mathbf{V}, \mathbf{W}\rangle \tag{3.48}$$

for every $\mathbf{W} \in D_f^3$. To approximate $\mathbf{X}$, we use a Raviert–Thomas finite-element subspace and solve (3.48) in the divergence-free subspace [16].

We remark that the solution of (3.48) in a divergence-free subspace will involve the approximate inversion of a mass matrix in this subspace. This can be easily accomplished by multilevel techniques [2].

4. Stokes-like regularity. In this section, we establish regularity estimates (3.34) for the Stokes-like equation (3.33). We use ADN theory [1] to prove that (3.34) applies when Ω has $C^{1,1}$ boundary.

Define operator $\mathcal{S}$ by

$$\mathcal{S}\mathbf{U} := \nabla\times\mathbf{U} - (\nabla\times\mathbf{U})^t, \tag{4.1}$$

that is,

$$\mathcal{S} = \nabla\times_{3\times 9} = \begin{pmatrix} 0 & 0 & 0 & -\partial_y & \partial_x & 0 & -\partial_z & 0 & \partial_x \\ \partial_y & -\partial_x & 0 & 0 & 0 & 0 & 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x & 0 & \partial_z & -\partial_y & 0 & 0 & 0 \end{pmatrix}. \tag{4.2}$$

Denote the operator $\nabla\cdot$ by $\mathcal{D}$, that is,

$$\mathcal{D} := \nabla\cdot_{3\times 9} = \begin{pmatrix} \partial_x & \partial_y & \partial_z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_x & \partial_y & \partial_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_x & \partial_y & \partial_z \end{pmatrix}. \tag{4.3}$$

Similarly define

$$\mathcal{G} := \nabla_{9\times 3} = (\nabla\cdot)^t = \mathcal{D}^t. \tag{4.4}$$

It is convenient to modify the Stokes-like system (3.33) by combining the interior equations $\nabla\cdot\mathbf{W} = \mathbf{0}$ and $\mathcal{S}\mathbf{W} = \mathbf{0}$, as well as the boundary equations $\nabla\cdot\mathbf{W} = \mathbf{0}$ and $\mathcal{S}\mathbf{W} = \mathbf{0}$. This we do by defining

$$\mathcal{S}_{\mathcal{D}} := \mathcal{S} + \mathcal{D} \tag{4.5}$$

and, for some $\alpha, \beta \in \Re$ to be determined later,

$$\mathcal{S}_{\mathcal{D}}^{\alpha,\beta} := \alpha\mathcal{S} + \beta\mathcal{D}. \tag{4.6}$$

Any solution of the system (3.33) will be a solution of

$$\left\{ \begin{array}{rcll} -\mathcal{R}\Delta\mathbf{W} + \nabla\times\mathcal{J}\boldsymbol{\eta} &=& \mathbf{F} & \text{in } \Omega, \\ \mathcal{S}_{\mathcal{D}}\mathbf{W} &=& \mathbf{0} & \text{in } \Omega, \\ \mathcal{S}_{\mathcal{D}}^{\alpha,\beta}\mathbf{W} &=& \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{n}\cdot\mathbf{W} &=& \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{n}\cdot\nabla\times\mathbf{W} &=& \mathbf{0} & \text{on } \partial\Omega. \end{array} \right. \tag{4.7}$$

This system involves the interior operator

$$\begin{pmatrix} -\mathcal{R}\Delta_{9\times 9} & \mathcal{C}_{9\times 3} \\ \mathcal{S}_{\mathcal{D}3\times 9} & \mathbf{0}_{3\times 3} \end{pmatrix}_{12\times 12}, \tag{4.8}$$

where $\mathcal{C}$ will be defined in (4.10) below, and the boundary operator

$$\begin{pmatrix} \mathcal{S}_{\mathcal{D}}^{\alpha,\beta}{}_{3\times 9} & \mathbf{0}_{3\times 3} \\ \mathbf{n}\cdot_{3\times 9} & \mathbf{0}_{3\times 3} \\ (\mathbf{n}\cdot\nabla\times)_{3\times 9} & \mathbf{0}_{3\times 3} \end{pmatrix}_{9\times 12}. \tag{4.9}$$

To coincide with our use of matrix and tensor product notation, now define the trace operator $\mathrm{tr}_{1\times 9}$ by

$$\mathrm{tr}_{1\times 9} = (1,0,0,0,1,0,0,0,1).$$

The next lemma will help in constructing $\mathcal{C}$.

LEMMA 4.1. *For* $\mathbf{N} \in \mathrm{Null}(A_2)$*, we have*

$$\mathcal{P}\left(\nabla\times\mathbf{N}\right) = \left(I_3 \otimes \nabla_{3\times 1} - \nabla\cdot_{1\times 3} \otimes \mathrm{tr}^t_{1\times 9}\right) \begin{pmatrix} r \\ q \\ p \end{pmatrix}.$$

Proof. Any $\mathbf{N} \in \mathrm{Null}(A_2)$ must be of the form $(0, p, -q, -p, 0, r, q, -r, 0)^t$, which implies that

$$\mathcal{P}(\nabla\times\mathbf{N}) = \begin{pmatrix} \nabla r \\ \nabla q \\ \nabla p \end{pmatrix} - \left[\nabla\cdot\begin{pmatrix} r \\ q \\ p \end{pmatrix}\right] \mathrm{tr}^t_{1\times 9},$$

which in turn proves the lemma. □

By Lemma 4.1 and the relation $\mathcal{P}^2 = I_9$, we have that

$$\mathcal{C} := \mathcal{P}\left(I_3 \otimes \nabla_{3\times 1} - \nabla\cdot_{1\times 3} \otimes \mathrm{tr}^t_{1\times 9}\right). \tag{4.10}$$

Note that the respective divergence and gradient operators can be written

$$\mathcal{D} = I_3 \otimes \nabla\cdot_{1\times 3}, \qquad \mathcal{G} = I_3 \otimes \nabla_{3\times 1}.$$

Let

$$\mathcal{H} := \nabla_{3\times 1} \otimes \mathrm{tr}_{1\times 9}.$$

Similarly, the operator $\mathcal{S}$ can be written

$$\mathcal{S} = \nabla_{3\times 1} \otimes \mathrm{tr}_{1\times 9} - \nabla\cdot_{1\times 3} \otimes I_3.$$

Let

$$\mathcal{T} := \nabla_{3\times 1} \otimes \mathrm{tr}_{1\times 9} - \nabla\cdot_{1\times 3} \otimes I_3 + I_3 \otimes \nabla\cdot_{1\times 3}.$$

The next lemma collects some simple but useful relations. We omit its proof since it consists of fairly straightforward algebra.

LEMMA 4.2. *The following operator relations hold:*

1. $\mathcal{D}\mathcal{P} = \nabla\cdot_{1\times 3} \otimes I_3$.
2. $\mathcal{D}^t = \mathcal{G}$.
3. $\mathcal{H}^t = \nabla\cdot_{1\times 3} \otimes \mathrm{tr}^t_{1\times 9}$.
4. $(I_9 - \mathcal{P})(I_9 + \mathcal{P}) = \mathbf{0}$.

We can now rewrite the interior operator (4.8) using (4.10):

$$\begin{pmatrix} -\left[\mathcal{R}\Delta\right]_{9\times 9} & \left[\mathcal{P}\left(I_3 \otimes \nabla_{3\times 1} - \nabla\cdot_{1\times 3} \otimes \mathrm{tr}^t_{1\times 9}\right)\right]_{9\times 3} \\ \left[\nabla_{3\times 1} \otimes \mathrm{tr}_{1\times 9} - \nabla\cdot_{1\times 3} \otimes I_3 + I_3 \otimes \nabla\cdot_{1\times 3}\right]_{3\times 9} & \mathbf{0}_{3\times 3} \end{pmatrix}. \tag{4.11}$$

Note that

$$\mathcal{S} = \mathcal{H} - \mathcal{D}\mathcal{P}, \quad \mathcal{T} = \mathcal{H} - \mathcal{D}\mathcal{P} + \mathcal{D}, \quad \text{and} \quad \mathcal{C} = \mathcal{P}\left(\mathcal{G} - \mathcal{H}^t\right). \tag{4.12}$$

ADN theory involves two main steps. The first step consists of assigning norms to each interior equation and unknown. The choices are guided by the aim of proving

that the system is *elliptic* or *strongly elliptic* in these norms. We use the notation developed in [1, Chapter I, section 1]. For system (3.33), we choose Sobolev norms based on the scales

$$s_1 = \cdots = s_9 = 0, \qquad s_{10} = s_{11} = s_{12} = -1 \tag{4.13}$$

for the equations and

$$t_1 = \cdots = t_9 = 2, \qquad t_{10} = t_{11} = t_{12} = 1 \tag{4.14}$$

for the variables. Next, we examine the principal part of the interior operator, which includes any term (i, j) whose order is $(t_i + s_j)$. In our case, no terms can be excluded, so the principal part is the entire operator. Next, we replace ∂_x, ∂_y, and ∂_z by ξ_1, ξ_2, and ξ_3, respectively, to form the corresponding stencil matrix $l(\boldsymbol{\xi})$ for (4.11):

$$l(\boldsymbol{\xi}) := \begin{pmatrix} -\rho\mathcal{R}_{9\times 9} & \hat{\mathcal{C}}_{9\times 3} \\ \hat{\mathcal{T}}_{3\times 9} & \mathbf{0}_{3\times 3} \end{pmatrix}, \quad \rho = |\boldsymbol{\xi}|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2, \tag{4.15}$$

where $\hat{\mathcal{C}}$ and $\hat{\mathcal{T}}$ are the respective representations of $\mathcal{C}$ and $\mathcal{T}$ in terms of $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^t$. From now on, we use the notation $\hat{\mathcal{M}}$ to represent a given matrix $\mathcal{M}$ in terms of $\boldsymbol{\xi}$.

As before, we omit the straightforward algebraic proof of our next lemma.

LEMMA 4.3. *The following stencil relations hold:*

1. $\hat{\mathcal{D}}^t = \hat{\mathcal{G}}$.
2. $\hat{\mathcal{H}}\mathcal{P} = \hat{\mathcal{H}}$.
3. $\hat{\mathcal{S}}\hat{\mathcal{G}} = \mathbf{0}$.
4. $\hat{\mathcal{H}}^t(\hat{\mathcal{D}} - \hat{\mathcal{D}}\mathcal{P}) = (\hat{\mathcal{H}}^t\hat{\mathcal{D}})(I_9 - \mathcal{P}) = \mathbf{0}$.
5. $\hat{\mathcal{H}}\hat{\mathcal{G}} = \frac{1}{3}\hat{\mathcal{H}}\hat{\mathcal{H}}^t = \hat{\mathcal{D}}\mathcal{P}\hat{\mathcal{G}} = \frac{1}{2}\hat{\mathcal{S}}\hat{\mathcal{H}}^t = \hat{\mathcal{D}}\hat{\mathcal{H}}^t = \boldsymbol{\xi}\boldsymbol{\xi}^t$.
6. $\hat{\mathcal{D}}\hat{\mathcal{G}} = \rho I_3$.
7. $\hat{\mathcal{H}}^t\hat{\mathcal{H}} = \rho \mathrm{tr}_{1\times 9}^t \mathrm{tr}_{1\times 9}$.
8. $\boldsymbol{\xi}^t(\hat{\mathcal{D}} - \hat{\mathcal{D}}\mathcal{P}) = \mathbf{0}$.
9. $\boldsymbol{\xi}\boldsymbol{\xi}^t\hat{\mathcal{H}} = \rho\hat{\mathcal{H}}$.
10. $\hat{\mathcal{S}}\mathcal{P}\hat{\mathcal{G}} = \boldsymbol{\xi}\boldsymbol{\xi}^t - \rho$.

Note that

$$\hat{\mathcal{C}} = \mathcal{P}\hat{\mathcal{G}} - \hat{\mathcal{H}}^t,$$

$$\mathcal{R}^{-1} = \frac{1}{c(c-2d)}\left[(c-d)I_9 - d\mathcal{P}\right], \tag{4.16}$$

$$\mathcal{R}^{-1}\hat{\mathcal{C}} = \frac{1}{c(c-2d)}\left[(c-d)\mathcal{P}\hat{\mathcal{G}} - d\hat{\mathcal{G}} - (c-2d)\hat{\mathcal{H}}^t\right], \tag{4.17}$$

and

$$\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}} = \frac{1}{c^2 - 2cd}\left(\hat{\mathcal{H}} - \hat{\mathcal{D}}\mathcal{P} + \hat{\mathcal{D}}\right)((c-d)I_9 - d\mathcal{P})\left(\mathcal{P}\hat{\mathcal{G}} - \hat{\mathcal{H}}^t\right). \tag{4.18}$$

We must now take the determinant $L(\boldsymbol{\xi}) := \det(\, l(\boldsymbol{\xi})\,)$.

PROPOSITION 4.1. *System* (3.33) *with* (4.13) *and* (4.14) *yields the determinant*

$$L(\boldsymbol{\xi}) = \tilde{c}\rho^9,$$

where $\tilde{c}$ is a nonzero constant, so it is elliptic of order 18 *in the ADN sense.*

Proof. One step of Gaussian elimination yields the block LU decomposition

$$l(\boldsymbol{\xi}) = \begin{pmatrix} I_9 & \mathbf{0} \\ -\frac{1}{\rho}\hat{\mathcal{T}}\mathcal{R}^{-1} & I_3 \end{pmatrix} \begin{pmatrix} -\rho\mathcal{R} & \hat{\mathcal{C}} \\ \mathbf{0} & \frac{1}{\rho}\left[\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right] \end{pmatrix}. \tag{4.19}$$

Hence,

$$\det(\, l(\boldsymbol{\xi})) = \rho^9 \det(\mathcal{R}) \det\left(\frac{1}{\rho}\left[\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right]\right). \tag{4.20}$$

But Lemma 4.3 implies that

$$\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}} = \frac{1}{c}\left[\hat{\mathcal{H}}\hat{\mathcal{G}} - \hat{\mathcal{H}}\hat{\mathcal{H}}^t\right] - \frac{1}{(c-2d)}\hat{\mathcal{D}}(I-\mathcal{P})\hat{\mathcal{G}}. \tag{4.21}$$

Hence,

$$-\frac{1}{\rho}\left[\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right] = \frac{1}{c-2d}\left[I + \frac{c-4d}{c}\frac{\boldsymbol{\xi}\boldsymbol{\xi}^t}{\rho}\right], \tag{4.22}$$

which has eigenvalues

$$\frac{2}{c}, \quad \frac{1}{c-2d}, \quad \text{and} \quad \frac{1}{c-2d}.$$

Thus,

$$\det\left[\frac{1}{\rho}\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right] = \frac{-2}{c(c-2d)^2},$$

which with (4.20) implies that

$$L(\boldsymbol{\xi}) = \det(-\rho\mathcal{R}) \det\left(\frac{1}{\rho}\left[\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right]\right) = \tilde{c}\rho^9 \quad (\tilde{c} \neq 0). \qquad \square$$

The next step in ADN theory involves the boundary. We first examine the so-called *supplementing* boundary condition by defining the polynomial

$$P(\kappa) := L(\boldsymbol{\xi} + \kappa\hat{\boldsymbol{\xi}}) = |\boldsymbol{\xi} + \kappa\hat{\boldsymbol{\xi}}|^{18},$$

where $\boldsymbol{\xi}$ and $\hat{\boldsymbol{\xi}}$ are any linearly independent vectors.

PROPOSITION 4.2. *$P(\kappa)$ has exactly nine roots with positive imaginary parts.*

Proof. Setting $|\boldsymbol{\xi} + \kappa\hat{\boldsymbol{\xi}}|^2 = |\boldsymbol{\xi}|^2 + 2\langle\boldsymbol{\xi}, \hat{\boldsymbol{\xi}}\rangle\kappa + \kappa^2|\hat{\boldsymbol{\xi}}|^2 = 0$ yields

$$\kappa = -\langle\boldsymbol{\xi}, \hat{\boldsymbol{\xi}}\rangle \pm \sqrt{\langle\boldsymbol{\xi}, \hat{\boldsymbol{\xi}}\rangle^2 - |\boldsymbol{\xi}|^2|\hat{\boldsymbol{\xi}}|^2}.$$

But $\langle\boldsymbol{\xi}, \hat{\boldsymbol{\xi}}\rangle - |\boldsymbol{\xi}||\hat{\boldsymbol{\xi}}| < 0$ because $\boldsymbol{\xi}$ and $\hat{\boldsymbol{\xi}}$ are linearly independent, which proves the proposition. $\square$

We next examine the so-called *complementing* boundary condition in an attempt to show that the boundary conditions of the system are *independent* of its interior equations and provide a well posed problem. First, form the stencil of the boundary operator (4.9), calling it $\mathcal{B}(\mathbf{x},\boldsymbol{\xi})$, where $\mathbf{x}\in\Omega$ and $\boldsymbol{\xi}=\boldsymbol{\tau}+\gamma\mathbf{n}$, with $\mathbf{n}$ the outward unit normal on $\partial\Omega$ and $\boldsymbol{\tau}$ any tangential unit vector on $\partial\Omega$. To represent an orthogonal tangent direction, define

$$\boldsymbol{\tau}^{\perp} := \mathbf{n}\times\boldsymbol{\tau}$$

on $\partial\Omega$. The stencil of the *curl* becomes

$$\mathcal{C}(\boldsymbol{\xi}) := \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}. \tag{4.23}$$

Note that

$$\mathbf{n}\cdot\nabla\times = \mathbf{n}\cdot\mathcal{C}(\boldsymbol{\xi}) = (\mathbf{n}\times\boldsymbol{\xi})^t = (\mathbf{n}\times(\boldsymbol{\tau}+\gamma\mathbf{n}))^t = (\boldsymbol{\tau}^{\perp})^t.$$

Then the operators $\mathbf{n}\cdot$ and $\mathbf{n}\cdot\nabla\times$ in (4.9) are, in fact,

$$\mathbf{n}\cdot = I_3\otimes\mathbf{n}^t, \qquad \mathbf{n}\cdot\nabla\times = I_3\otimes(\boldsymbol{\tau}^{\perp})^t.$$

Thus, boundary operator (4.9) has the stencil

$$\hat{\mathcal{B}}(\mathbf{x},\boldsymbol{\xi}) = \begin{pmatrix} \hat{\mathcal{S}}_{\mathcal{D}}^{\alpha,\beta}{}_{3\times 9} & \mathbf{0} \\ I_3\otimes\mathbf{n}^t & \mathbf{0} \\ I_3\otimes(\boldsymbol{\tau}^{\perp})^t & \mathbf{0} \end{pmatrix}_{9\times 12}. \tag{4.24}$$

We examine linear independence of the rows of

$$\hat{\mathcal{B}}\hat{\mathcal{L}} := \hat{\mathcal{B}}(\mathbf{x},\boldsymbol{\xi})\,\mathrm{adj}\,(l(\boldsymbol{\xi})) = \mathcal{L}(\boldsymbol{\xi})\hat{\mathcal{B}}(\mathbf{x},\boldsymbol{\xi})l^{-1}(\boldsymbol{\xi}) = \tilde{c}\rho^9\hat{\mathcal{B}}(\mathbf{x},\boldsymbol{\xi})l^{-1}(\boldsymbol{\xi}). \tag{4.25}$$

Note that (4.22) and Lemma 4.3 yield

$$\left[\frac{1}{\rho}\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right]^{-1} = -(c-2d)\left[I_3 - \frac{c-4d}{2(c-2d)}\frac{\boldsymbol{\xi}\boldsymbol{\xi}^t}{\rho}\right]. \tag{4.26}$$

LEMMA 4.4. *The inverse of* $l(\boldsymbol{\xi})$ *is*

$$l^{-1}(\boldsymbol{\xi}) = \begin{pmatrix} \frac{-1}{\rho}\mathcal{R}^{-1} + \frac{1}{\rho^2}\mathcal{R}^{-1}\hat{\mathcal{C}}\left[\frac{1}{\rho}\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right]^{-1}\hat{\mathcal{T}}\mathcal{R}^{-1} & \frac{1}{\rho}\mathcal{R}^{-1}\hat{\mathcal{C}}\left[\frac{1}{\rho}\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right]^{-1} \\ \frac{1}{\rho}\left[\frac{1}{\rho}\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right]^{-1}\hat{\mathcal{T}}\mathcal{R}^{-1} & \left[\frac{1}{\rho}\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right]^{-1} \end{pmatrix}. \tag{4.27}$$

Proof. A lengthy but straightforward calculation yields the conclusion. ∎

Linear independence of the rows of matrix $\mathcal{BL}$ is equivalent to that of the rows of

$$\widehat{\mathcal{BL}} := \frac{1}{\tilde{c}\rho^8}\left(\hat{\mathcal{B}}\hat{\mathcal{L}}\right)_{9\times 12}\begin{pmatrix}\mathcal{R} & \mathbf{0}\\ \mathbf{0} & I_3\end{pmatrix}_{12\times 12} \tag{4.28}$$

$$= \begin{pmatrix} \hat{\mathcal{S}}_{\mathcal{D}}^{\alpha,\beta}\left[-I_9 + \frac{1}{\rho}\mathcal{R}^{-1}\hat{\mathcal{C}}\left[\frac{1}{\rho}\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right]^{-1}\hat{\mathcal{T}}\right] & * \\ (I_3\otimes\mathbf{n}^t)\left[-I_9 + \frac{1}{\rho}\mathcal{R}^{-1}\hat{\mathcal{C}}\left[\frac{1}{\rho}\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right]^{-1}\hat{\mathcal{T}}\right] & * \\ \left(I_3\otimes(\boldsymbol{\tau}^{\perp})^t\right)\left[-I_9 + \frac{1}{\rho}\mathcal{R}^{-1}\hat{\mathcal{C}}\left[\frac{1}{\rho}\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right]^{-1}\hat{\mathcal{T}}\right] & * \end{pmatrix},$$

where $*$ stands for a matrix that does not affect row independence (as we will see).

Let

$$\delta := \frac{d}{c-2d}. \tag{4.29}$$

Then (4.17) becomes

$$\mathcal{R}^{-1}\hat{\mathcal{C}} = -\frac{1}{c}\left[\hat{\mathcal{H}}^t + \delta\hat{\mathcal{G}} - (1+\delta)\mathcal{P}\hat{\mathcal{G}}\right]. \tag{4.30}$$

We view (4.28) as a 3×2 block matrix.

LEMMA 4.5. *The* $(1,1)$ *block of* $\widehat{\mathcal{BL}}$ *is*

$$(\alpha-\beta)\left(\frac{1+\delta}{1+2\delta}\hat{\mathcal{D}} + \frac{\delta}{1+2\delta}\hat{\mathcal{D}}\mathcal{P}\right). \tag{4.31}$$

Proof. Combining (4.30) with Lemma 4.3 yields

$$\hat{\mathcal{S}}_{\mathcal{D}}^{\alpha\beta}\mathcal{R}^{-1}\hat{\mathcal{C}} = \left(-\frac{1}{c}\right)\left[\rho[(1+\delta)\alpha+\beta\delta]I_3 + [\alpha(1-\delta)-\beta\delta]\boldsymbol{\xi}\boldsymbol{\xi}^t\right],$$

and combining (4.12), (4.26), and (4.29) with Lemma 4.3 yields

$$\left[\frac{1}{\rho}\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right]^{-1}\hat{\mathcal{T}} = (-c)\left[\frac{1}{2}\hat{\mathcal{H}} + \frac{1}{1+2\delta}[\hat{\mathcal{D}}-\hat{\mathcal{D}}\mathcal{P}]\right]. \tag{4.32}$$

The conclusion thus follows via a little algebra. ∎

REMARK 4.1. *Because of Lemma* 4.5, *we may choose any* α *and* β *in* $\hat{\mathcal{S}}_{\mathcal{D}}^{\alpha\beta}$ *such that*

$$\alpha-\beta\neq 0,$$

so we may take $\alpha=0$ *and* $\beta=-1$. *Then the* $(1,1)$ *block of* $\widehat{\mathcal{BL}}$ *becomes*

$$\frac{1}{1+2\delta}\left[(1+\delta)\hat{\mathcal{D}} + \delta\hat{\mathcal{D}}\mathcal{P}\right]. \tag{4.33}$$

We state the following lemma, again omitting the straightforward proof.

LEMMA 4.6. *The following matrix relations hold:*

1. $(I_3 \otimes \mathbf{n}^t)\hat{\mathcal{H}}^t = \mathbf{n}\boldsymbol{\xi}^t$.
2. $(I_3 \otimes \mathbf{n}^t)\hat{\mathcal{G}} = (\mathbf{n}\cdot\boldsymbol{\xi})I_3$.
3. $(I_3 \otimes \mathbf{n}^t)(\mathcal{P}\hat{\mathcal{G}}) = \boldsymbol{\xi}\mathbf{n}^t$.
4. $\hat{\mathcal{H}} = \boldsymbol{\xi}\mathrm{tr}_{1\times 9} = \boldsymbol{\xi}\mathbf{b}^t$.
5. $\hat{\mathcal{D}} - \hat{\mathcal{D}}\mathcal{P} = I_3 \otimes \boldsymbol{\xi}^t - \boldsymbol{\xi}^t \otimes I_3$.
6. $\hat{\mathcal{D}} = I_3 \otimes \boldsymbol{\xi}^t$.
7. $\mathbf{n}\boldsymbol{\xi}^t(I_3 \otimes \boldsymbol{\xi}^t - \boldsymbol{\xi}^t \otimes I_3) = \mathbf{0}$.
8. $\boldsymbol{\xi}\mathbf{n}^t(I_3 \otimes \boldsymbol{\xi}^t - \boldsymbol{\xi}^t \otimes I) = \boldsymbol{\xi}(\mathbf{n}^t \otimes \boldsymbol{\xi}^t - \boldsymbol{\xi}^t \otimes \mathbf{n}^t)$.

LEMMA 4.7. *The* $(2,1)$ *block of* $\widehat{\mathcal{BL}}$ *is*

$$-I_3\otimes\mathbf{n}^t+\frac{1}{\rho}\left[\frac{1}{2}\mathbf{n}\mathbf{b}^t - \frac{(\mathbf{n}\cdot\boldsymbol{\xi})}{2}\boldsymbol{\tau}\mathbf{b}^t + \frac{\delta(\mathbf{n}\cdot\boldsymbol{\xi})}{1+2\delta}\left(\hat{\mathcal{D}} - \hat{\mathcal{D}}\mathcal{P}\right) - \frac{1+\delta}{1+2\delta}\boldsymbol{\xi}\left(\mathbf{n}^t\otimes\boldsymbol{\xi}^t - \boldsymbol{\xi}^t\otimes\mathbf{n}^t\right)\right]. \tag{4.34}$$

Proof. Combining (4.30) and (4.32) with Lemma 4.6 yields the conclusion. ∎

We state the following lemma, again omitting the straightforward proof.

LEMMA 4.8. *The following tangent vector relations hold:*

1. $\boldsymbol{\tau}^\perp\boldsymbol{\xi}^t\left(I_3 \otimes \boldsymbol{\xi}^t - \boldsymbol{\xi}^t \otimes I_3\right) = \mathbf{0}$.
2. $[I_3 \otimes (\boldsymbol{\tau}^\perp)^t]\hat{\mathcal{H}}^t = \boldsymbol{\tau}^\perp\boldsymbol{\xi}^t$.
3. $[I_3 \otimes (\boldsymbol{\tau}^\perp)^t]\hat{\mathcal{G}} = (\boldsymbol{\tau}^\perp\cdot\boldsymbol{\xi})I_3 = \mathbf{0}$.
4. $[I_3 \otimes (\boldsymbol{\tau}^\perp)^t](\mathcal{P}\hat{\mathcal{G}}) = \boldsymbol{\xi}(\boldsymbol{\tau}^\perp)^t$.
5. $\boldsymbol{\tau}^\perp\cdot\boldsymbol{\xi} = (\boldsymbol{\tau}^\perp)^t\boldsymbol{\xi} = 0$.
6. $(\boldsymbol{\tau}^\perp)^t(I_3 \otimes \boldsymbol{\xi}^t - \boldsymbol{\xi}^t \otimes I_3) = (\boldsymbol{\tau}^\perp)^t \otimes \boldsymbol{\xi}^t - \boldsymbol{\xi}^t \otimes (\boldsymbol{\tau}^\perp)^t$.

LEMMA 4.9. *The* $(3,1)$ *block of* $\widehat{\mathcal{BL}}$ *is*

$$-I_3 \otimes (\boldsymbol{\tau}^\perp)^t + \frac{1}{\rho}\left[\frac{\rho}{2}\boldsymbol{\tau}^\perp\mathbf{b}^t - \frac{1+\delta}{1+2\delta}\boldsymbol{\xi}\left[(\boldsymbol{\tau}^\perp)^t\otimes\boldsymbol{\xi}^t - \boldsymbol{\xi}^t\otimes(\boldsymbol{\tau}^\perp)^t\right]\right]. \tag{4.35}$$

Proof. Note that (4.32) and Lemma 4.6 yield

$$\left(\frac{1}{\rho}\hat{\mathcal{T}}\mathcal{R}^{-1}\hat{\mathcal{C}}\right)^{-1}\hat{\mathcal{T}} = (-c)\left[\frac{1}{2}\boldsymbol{\xi}\mathbf{b}^t + \frac{1}{1+2\delta}\left(I_3\otimes\boldsymbol{\xi}^t - \boldsymbol{\xi}^t\otimes I_3\right)\right].$$

Moreover, (4.17) and Lemma 4.8 yield

$$\left(I_3\otimes(\boldsymbol{\tau}^\perp)^t\right)\mathcal{R}^{-1}\hat{\mathcal{C}} = -\frac{1}{c}\left(\boldsymbol{\tau}^\perp\boldsymbol{\xi}^t - (1+\delta)\boldsymbol{\xi}(\boldsymbol{\tau}^\perp)^t\right).$$

Thus, Lemma 4.8 again leads to the conclusion. ∎

Independence of the rows of $\widehat{\mathcal{BL}}$ is equivalent to that of

$$\widetilde{\mathcal{BL}} := \left(\widehat{\mathcal{BL}}\right)\begin{pmatrix}(1+\delta)I_9 - \delta\mathcal{P} & \mathbf{0}\\ \mathbf{0} & I_3\end{pmatrix}. \tag{4.36}$$

PROPOSITION 4.3. *We have the following:*

(1) *The* $(1,1)$ *block of* $\widetilde{\mathcal{BL}}$ *becomes* $\hat{\mathcal{D}}_{9\times 9}$.

(2) *The* $(2,1)$ *block of* $\widetilde{\mathcal{BL}}$ *becomes*

$$-I_3\otimes\mathbf{n}^t - \delta(I_3\otimes\mathbf{n}^t)(I_9-\mathcal{P}) + \frac{1}{\rho}\left[\frac{1}{2}(\mathbf{n}-(\mathbf{n}\cdot\boldsymbol{\xi})\boldsymbol{\tau})\mathbf{b}^t + \delta(\mathbf{n}\cdot\boldsymbol{\xi})(\hat{\mathcal{D}}-\hat{\mathcal{D}}\mathcal{P}) - (1+\delta)\boldsymbol{\xi}(\mathbf{n}^t\otimes\boldsymbol{\xi}^t)(I_9-\mathcal{P})\right].$$

(3) *The* $(3,1)$ *block of* $\widetilde{\mathcal{BL}}$ *becomes*

$$-I_3 \otimes (\boldsymbol{\tau}^\perp)^t - \delta(I_3 \otimes (\boldsymbol{\tau}^\perp)^t)(I_9 - \mathcal{P}) + \frac{1}{\rho}\left[\frac{\rho}{2}\boldsymbol{\tau}^\perp \mathbf{b}^t - (1+\delta)\boldsymbol{\xi}((\boldsymbol{\tau}^\perp)^t \otimes \boldsymbol{\xi}^t)(I_9 - \mathcal{P})\right].$$

Proof. The conclusions follow by multiplying (4.33), (4.34), and (4.35) on the right by $(1+\delta)I_9 - \delta\mathcal{P}$ and using the relations

$$\begin{aligned}
(I_3 \otimes \mathbf{n}^t)\left[(1+\delta)I_9 - \delta\mathcal{P}\right] &= I_3 \otimes \mathbf{n}^t + \delta(I_3 \otimes \mathbf{n}^t)(I_9 - \mathcal{P}),\\
\mathbf{b}^t\mathcal{P} &= \mathbf{b}^t\left((1+\delta)I_9 - \delta\mathcal{P}\right) = \mathbf{b}^t,\\
\left(\hat{\mathcal{D}} - \hat{\mathcal{D}}\mathcal{P}\right)\left((1+\delta)I_9 - \delta\mathcal{P}\right) &= (1+2\delta)(\hat{\mathcal{D}} - \hat{\mathcal{D}}\mathcal{P}),\\
\boldsymbol{\tau}\mathbf{b}^t\left((1+\delta)I_9 - \delta\mathcal{P}\right) &= \boldsymbol{\tau}\mathbf{b}^t,\\
(\mathbf{x}^t \otimes \mathbf{y}^t)\mathcal{P} &= \mathbf{y}^t \otimes \mathbf{x}^t,
\end{aligned}$$

where $\mathbf{x}$ and $\mathbf{y}$ $(\mathbf{x} \neq \mathbf{y})$ are taken from among $\mathbf{n}$, $\boldsymbol{\xi}$, and $\boldsymbol{\tau}^\perp$. ∎

As a result of this proposition, $\widetilde{\mathcal{BL}}$ viewed as a 3×2 block matrix becomes

$$\widetilde{\mathcal{BL}} = \frac{1}{\rho}\begin{pmatrix} \rho\hat{\mathcal{D}} & * \\ -\rho\left[I_3 \otimes \mathbf{n}^t + \delta(I_3 \otimes \mathbf{n}^t)(I_9 - \mathcal{P})\right] & \\ +\frac{1}{2}(\mathbf{n} - (\mathbf{n}\cdot\boldsymbol{\xi})\boldsymbol{\tau})\mathbf{b}^t + \delta(\mathbf{n}\cdot\boldsymbol{\xi})(\hat{\mathcal{D}} - \hat{\mathcal{D}}\mathcal{P}) & \\ -(1+\delta)\boldsymbol{\xi}(\mathbf{n}^t \otimes \boldsymbol{\xi}^t)(I_9 - \mathcal{P}) & * \\ -\rho\left[I_3 \otimes (\boldsymbol{\tau}^\perp)^t + \delta(I_3 \otimes (\boldsymbol{\tau}^\perp)^t)(I_9 - \mathcal{P})\right] & \\ +\frac{1}{2}\rho\boldsymbol{\tau}^\perp\mathbf{b}^t - (1+\delta)\boldsymbol{\xi}((\boldsymbol{\tau}^\perp)^t \otimes \boldsymbol{\xi}^t)(I_9 - \mathcal{P}) & * \end{pmatrix}, \tag{4.37}$$

where $*$ again stands for a matrix that does not affect row independence.

Recalling that

$$\rho = 1 + \gamma^2, \quad \boldsymbol{\xi} = \boldsymbol{\tau} + \gamma\mathbf{n}, \quad \text{and} \quad \mathbf{n}\cdot\boldsymbol{\xi} = \gamma,$$

we can rewrite the three blocks of $\widetilde{\mathcal{BL}}$ in terms of γ. Because

$$\begin{aligned}
\rho\hat{\mathcal{D}} &= I_3 \otimes \left(\boldsymbol{\tau}^t + \gamma\mathbf{n}^t + \gamma^2\boldsymbol{\tau}^t + \gamma^3\mathbf{n}^t\right),\\
\hat{\mathcal{D}} - \hat{\mathcal{D}}\mathcal{P} &= \left(I_3 \otimes \boldsymbol{\tau}^t - \boldsymbol{\tau}^t \otimes I_3\right) + \gamma\left(I_3 \otimes \mathbf{n}^t - \mathbf{n}^t \otimes I_3\right),\\
\boldsymbol{\xi}\left(\mathbf{n}^t \otimes \boldsymbol{\xi}^t\right) &= \boldsymbol{\tau}\left(\mathbf{n}^t \otimes \boldsymbol{\tau}^t\right) + \gamma\left[\mathbf{n}\left(\mathbf{n}^t \otimes \boldsymbol{\tau}^t\right) + \boldsymbol{\tau}\left(\mathbf{n}^t \otimes \mathbf{n}^t\right)\right] + \gamma^2\left[\mathbf{n}\left(\mathbf{n}^t \otimes \mathbf{n}^t\right)\right],\\
\boldsymbol{\xi}\left(\mathbf{n}^t \otimes \boldsymbol{\xi}^t\right)(I_9 - \mathcal{P}) &= \boldsymbol{\tau}\left(\mathbf{n}^t \otimes \boldsymbol{\tau}^t - \boldsymbol{\tau}^t \otimes \mathbf{n}^t\right) + \gamma\mathbf{n}\left(\mathbf{n}^t \otimes \boldsymbol{\tau}^t - \boldsymbol{\tau}^t \otimes \mathbf{n}^t\right),
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\xi}\left((\boldsymbol{\tau}^\perp)^t \otimes \boldsymbol{\xi}^t\right)(I_9 - \mathcal{P}) = \boldsymbol{\tau}&\left[(\boldsymbol{\tau}^\perp)^t \otimes \boldsymbol{\tau}^t - \boldsymbol{\tau}^t \otimes (\boldsymbol{\tau}^\perp)^t\right]\\
&+ \gamma\left[\boldsymbol{\tau}\left((\boldsymbol{\tau}^\perp)^t \otimes \mathbf{n}^t - \mathbf{n}^t \otimes (\boldsymbol{\tau}^\perp)^t\right)\right.\\
&+ \left.\mathbf{n}\left((\boldsymbol{\tau}^\perp)^t \otimes \boldsymbol{\tau}^t - \boldsymbol{\tau}^t \otimes (\boldsymbol{\tau}^\perp)^t\right)\right]\\
&+ \gamma^2\mathbf{n}\left[(\boldsymbol{\tau}^\perp)^t \otimes \mathbf{n}^t - \mathbf{n}^t \otimes (\boldsymbol{\tau}^\perp)^t\right],
\end{aligned}$$

we can rewrite $\widetilde{\mathcal{BL}}$ as a matrix polynomial of γ:

$$\widetilde{\mathcal{BL}} = \frac{1}{(1+\gamma^2)}\left(\mathcal{A}_0 + \gamma\mathcal{A}_1 + \gamma^2\mathcal{A}_2 + \gamma^3\mathcal{A}_3, \quad \mathcal{A}_{j,4}\right), \tag{4.38}$$

where $\mathcal{A}_{j,4}$ is the matrix in the $(j,2)$ block $(j=1,2,3)$ of $\widetilde{\mathcal{BL}}$ in (4.37),

$$\mathcal{A}_0 := \begin{pmatrix} I_3 \otimes \boldsymbol{\tau}^t \\ -\left[I_3 \otimes \mathbf{n}^t + \delta(I_3 \otimes \mathbf{n}^t)(I_9 - \mathcal{P})\right] \\ +\frac{1}{2}\mathbf{n}\mathbf{b}^t - (1+\delta)\boldsymbol{\tau}\left(\mathbf{n}^t \otimes \boldsymbol{\tau}^t - \boldsymbol{\tau}^t \otimes \mathbf{n}^t\right) \\ -\left[I_3 \otimes (\boldsymbol{\tau}^\perp)^t + \delta(I_3 \otimes (\boldsymbol{\tau}^\perp)^t)(I_9 - \mathcal{P})\right] + \frac{1}{2}\boldsymbol{\tau}^\perp\mathbf{b}^t \\ -(1+\delta)\boldsymbol{\tau}\left[(\boldsymbol{\tau}^\perp)^t \otimes \boldsymbol{\tau}^t - \boldsymbol{\tau}^t \otimes (\boldsymbol{\tau}^\perp)^t\right] \end{pmatrix}_{9\times 9}, \tag{4.39}$$

$$\mathcal{A}_1 := \begin{pmatrix} I_3 \otimes \mathbf{n}^t \\ -\frac{1}{2}\boldsymbol{\tau}\mathbf{b}^t + \delta\left[I_3 \otimes \boldsymbol{\tau}^t - \boldsymbol{\tau}^t \otimes I_3\right] - (1+\delta)\mathbf{n}\left[\mathbf{n}^t \otimes \boldsymbol{\tau}^t - \boldsymbol{\tau}^t \otimes \mathbf{n}^t\right] \\ -(1+\delta)\left[\boldsymbol{\tau}\left((\boldsymbol{\tau}^\perp)^t \otimes \mathbf{n}^t - \mathbf{n}^t \otimes (\boldsymbol{\tau}^\perp)^t\right) + \mathbf{n}\left((\boldsymbol{\tau}^\perp)^t \otimes \boldsymbol{\tau}^t - \boldsymbol{\tau}^t \otimes (\boldsymbol{\tau}^\perp)^t\right)\right] \end{pmatrix}_{9\times 9}, \tag{4.40}$$

$$\mathcal{A}_2 := \begin{pmatrix} I_3 \otimes \boldsymbol{\tau}^t \\ -I_3 \otimes \mathbf{n}^t \\ -\left[I_3 \otimes (\boldsymbol{\tau}^\perp)^t + \delta(I_3 \otimes (\boldsymbol{\tau}^\perp)^t)(I_9 - \mathcal{P})\right] + \frac{1}{2}\boldsymbol{\tau}^\perp\mathbf{b}^t \\ -(1+\delta)\mathbf{n}\left[(\boldsymbol{\tau}^\perp)^t \otimes \mathbf{n}^t - \mathbf{n}^t \otimes (\boldsymbol{\tau}^\perp)^t\right] \end{pmatrix}_{9\times 9}, \tag{4.41}$$

and

$$\mathcal{A}_3 := \begin{pmatrix} I_3 \otimes \mathbf{n}^t \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}_{9\times 9}. \tag{4.42}$$

Note that the only root of

$$\mathcal{L}(\boldsymbol{\tau} + \gamma\mathbf{n}) = |\boldsymbol{\tau} + \gamma\mathbf{n}|^{18} = (1+\gamma^2)^9 = 0$$

with positive imaginary part is i with multiplicity 9. Let

$$M^+(\boldsymbol{\tau}, \mathbf{n}) = (\gamma - i)^9.$$

PROPOSITION 4.4. *The rows of $\widetilde{\mathcal{BL}}$ are linearly independent in the following sense:*

$$\sum_{h=1}^{9} c_h \left(\widetilde{\mathcal{BL}}\right)_{h,k} \equiv \mathbf{0} \bmod \left(M^+(\boldsymbol{\tau}, \mathbf{n})\right) \quad \textit{for} \;\; k = 1, \ldots, 12 \qquad \textit{implies} \;\; c_h = 0 \quad \forall\, h.$$

Proof. It is enough to show that, for any polynomial $\mathbf{p}_1(\gamma) = \mathcal{E}\gamma + \mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ are 9×1 matrices, the only solution of

$$\left(\mathcal{A}_0^t + \gamma\mathcal{A}_1^t + \gamma^2\mathcal{A}_2^t + \gamma^3\mathcal{A}_3^t\right)\mathbf{C} = \mathbf{p}_1(\gamma)\,(\gamma - i)^2 \tag{4.43}$$

is

$$\mathbf{C} = \mathbf{0}.$$

Expanding the right-hand side of (4.43) and equating terms with equal powers of γ imply

$$\mathcal{E} = \mathcal{A}_3^t\mathbf{C}, \quad \mathcal{F} - 2i\mathcal{E} = \mathcal{A}_2^t\mathbf{C}, \quad -\mathcal{E} - 2i\mathcal{F} = \mathcal{A}_1^t\mathbf{C}, \quad -\mathcal{F} = \mathcal{A}_0^t\mathbf{C},$$

so that

$$\mathcal{F} - 2i\mathcal{E} = -\left(\mathcal{A}_0^t + 2i\mathcal{A}_3^t\right)\mathbf{C}, \qquad -\mathcal{E} - 2i\mathcal{F} = \left(2i\mathcal{A}_0^t - \mathcal{A}_3^t\right)\mathbf{C}.$$

Then $\mathbf{C}$ must satisfy

$$\left(2i\mathcal{A}_3^t + \mathcal{A}_2^t + \mathcal{A}_0^t\right)\mathbf{C} = \mathbf{0}, \quad \left(\mathcal{A}_3^t + \mathcal{A}_1^t - 2i\mathcal{A}_0^t\right)\mathbf{C} = \mathbf{0}.$$

Multiplying the first equation above by $2i$ and adding to the second equation yields

$$\left(-3\mathcal{A}_3^t + 2i\mathcal{A}_2^t + \mathcal{A}_1^t\right)_{9\times 9}\mathbf{C} = \mathbf{0}. \tag{4.44}$$

Without loss of generality, we may change the basis from $\{\mathbf{n}, \boldsymbol{\tau}, \boldsymbol{\tau}^\perp\}$ to $\{(1,0,0)^t, (0,1,0)^t, (0,0,1)^t\}$, that is, we may take $\mathbf{n} = (1,0,0)^t$, $\boldsymbol{\tau} = (0,0,1)^t$, and $\boldsymbol{\tau}^\perp = (0,0,1)^t$ in (4.44). Define $\mathcal{B} := (\mathbf{n}, \boldsymbol{\tau}, \boldsymbol{\tau}^\perp)_{3\times 3}$. The change of basis can be accomplished by forming

$$\left(I_3 \otimes \mathcal{B}^t\right)\left(-3\mathcal{A}_3^t + 2i\mathcal{A}_2^t + \mathcal{A}_1^t\right)_{9\times 9}\left(I_3 \times \mathcal{B}\right).$$

Then the coefficient matrix in (4.44) becomes

$$-3\mathcal{A}_3^t + 2i\mathcal{A}_2^t + \mathcal{A}_1^t = \begin{pmatrix} -2 & 0 & 0 & -2i & -\frac{1}{2} & 0 & 0 & 0 & i \\ 2i & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+\delta & 0 \\ 0 & -2 & 0 & 1 & -2i & 0 & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & -\delta & 1+\delta & -2i(1+\delta) & 0 \\ 0 & 0 & -2 & 0 & 0 & -2i & -2i & -(1+\delta) & 0 \\ 0 & 0 & 2i & 0 & 0 & \delta & -(1+\delta) & 2i\delta & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -i \end{pmatrix}.$$

Gaussian elimination shows that $\det(-3\mathcal{A}_3^t + 2i\mathcal{A}_2^t + \mathcal{A}_1^t) \neq 0$, which proves the proposition. □

If we now appeal to Theorem 10.5 and the Remark on page 78 of [1], then the above results together with the uniqueness of the solution guaranteed by (4.7) prove the following.

THEOREM 4.1. *Let $\mathbf{F} \in L^2(\Omega)^9$ and assume that Ω has $C^{1,1}$ boundary. Then the unique solution $(\mathbf{W}, \boldsymbol{\eta}) \in H^2(\Omega)^9 \times (H^1(\Omega)^3 \cap L_0^2(\Omega)^3)$ of (3.33) satisfies*

$$\|\mathbf{W}\|_2^2 + \|\boldsymbol{\eta}\|_1^2 \leq C\|\mathbf{F}\|^2,$$

for some constant C depending only on Ω.

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