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FIRST-ORDER SYSTEM LEAST SQUARES FOR SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS: PART II*

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Abstract. This paper develops a least-squares functional that arises from recasting general second-order uniformly elliptic partial differential equations in $n = 2$ or 3 dimensions as a system of first-order equations. In part I [Z. Cai, R. D. Lazarov, T. Manteuffel, and S. McCormick, *SIAM J. Numer. Anal.*, 31 (1994), pp. 1785–1799] a similar functional was developed and shown to be elliptic in the $H(\mathrm{div}) \times H^1$ norm and to yield optimal convergence for finite element subspaces of $H(\mathrm{div}) \times H^1$. In this paper the functional is modified by adding a compatible constraint and imposing additional boundary conditions on the first-order system. The resulting functional is proved to be elliptic in the $(H^1)^{n+1}$ norm. This immediately implies optimal error estimates for finite element approximation by standard subspaces of $(H^1)^{n+1}$. Another direct consequence of this ellipticity is that multiplicative and additive multigrid algorithms applied to the resulting discrete functionals are optimally convergent. As an alternative to perturbation-based approaches, the least-squares approach developed here applies directly to convection–diffusion–reaction equations in a unified way and also admits a fast multigrid solver, historically a missing ingredient in least-squares methodology.

Key words. least-squares discretization, multigrid, second-order elliptic problems, iterative methods

AMS subject classifications. 65F10, 65F30

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1. Introduction.

The object of study of this paper, and its earlier companion [11], is the solution of elliptic equations (including convection–diffusion and Helmholtz equations) by way of a least-squares formulation for an equivalent first-order system. Such formulations have been considered by several researchers over the last few decades (see the historical discussion in [11]), motivated in part by the possibility of a well-posed variational principle for a general class of problems. In [11] a similar functional was developed and shown to be elliptic in the $H(\mathrm{div}) \times H^1$ norm and to yield optimal convergence for finite element subspaces of $H(\mathrm{div}) \times H^1$. In this paper the functional is modified by adding a compatible constraint and imposing additional boundary conditions on the first-order system. It is shown that the resulting functional is elliptic in the $(H^1)^{n+1}$ norm. Direct consequences of this result are optimal approximation error estimates for standard finite element subspaces of $(H^1)^{n+1}$ and optimal convergence of multiplicative and additive multigrid algorithms applied to the resulting discrete functionals. As an alternative to perturbation-based approaches (cf. [1, 3, 9, 10, 25, 34, 35]), the least-squares approach developed here applies directly to convection–diffusion–reaction equations in a unified way and also admits an efficient multilevel solver, historically a missing ingredient in least-squares methodology.

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The least-squares formulation considered in this paper differs from that of [11] (see also [31]) in that it incorporates a curl-free constraint and tangential boundary conditions on $\mathbf{u}$ (see section 2). Like the original in [11], our modified least-squares formulation avoids the so-called inf-sup condition of Ladyzhenskaya, Babŭska, and Brezzi (see [8]) and its attendant restrictions on the choice of finite element approximation subspaces. However, our modifications to the original form, while unnecessary for discretization accuracy (see [11]), greatly simplify the solution process. The unmodified form must be handled carefully because oscillatory divergence-free error components give relatively small residuals; such components must be specially treated in the multigrid relaxation process or eliminated in the discretization, for example. However, our incorporation of the curl-free constraint here exposes these components: in fact, *all* oscillatory error results in relatively large residuals. The modified functional is easily seen to be equivalent to a modified form of the $(H(\mathrm{div})\cap H(\mathbf{curl}))\times H^1$ norm. This norm may be sufficient to yield optimal finite element convergence if care is taken in choosing the finite element spaces [33, 27]. In addition, optimal multigrid convergence can be guaranteed under certain additional constraints [27]. The theory involving this norm is still incomplete.

In this paper the modified $(H(\mathrm{div}) \cap H(\mathbf{curl})) \times H^1$ norm is shown to be equivalent to the $(H^1)^{n+1}$ norm under some additional regularity assumptions. Thus, the modified functional yields full H^1-norm equivalence for each of the system variables. This comes at some loss of generality of the results presented here in that our proof requires H^2 regularity of the original problem with the lower-order terms removed. This should be expected, however, since our aim here is to obtain H^1-norm discretization error bounds on the fluxes.

For problems with reduced regularity, alternate forms of the least-squares functional must be considered, with the specific choice of form dependent on the goal of computation: for $H(\mathrm{div})$-type flux estimates, the original functional in [11] is appropriate; for weaker L^2-type flux estimates, for example when the right-hand side is in H^{-1}, an H^{-1} or mesh-weighted norm approach can be used (cf. [2, 4]); and for stronger local H^1-type flux estimates, for example when the coefficients are discontinuous, an appropriate local H^1 approach can be used [27].

The basic idea of the approach here is simple: the original functional in [11], which fails to be fully $(H^1)^{n+1}$ elliptic because it incorporates only a divergence-type flux derivative, is augmented by an admissible curl-type term. It is therefore immediate that the new functional is equivalent to an $(H(\mathrm{div}) \cap H(\mathbf{curl})) \times H^1$-type norm. However, to show that such a norm is in fact an $(H^1)^{n+1}$ norm is another matter: for that (and only that!) we need H^2 regularity of the original scalar problem without the lower-order terms, and we need to carefully extend basic results of functional analysis to account for general diffusion tensors and boundary conditions. This is the principal theoretical contribution of this paper.

The idea of adding the curl constraint in developing a least-squares functional for Poisson's equation has been used by several researchers (see, e.g., [13, 16, 17, 23, 14, 15, 24, 30]). For Poisson's equation, the key tool is the proof that $(H(\mathrm{div})\cap H(\mathbf{curl}))$ is algebraically and topologically imbedded in $(H^1)^n$, which was developed by Girault and Raviart [20] for problems with strictly Dirichlet or Neumann boundary conditions. Here, we extend the result in [20] to the context of a general diffusion tensor and a variety of boundary conditions.

The fundamental goal of our least-squares work is to develop a functional that is fully $(H^1)^{n+1}$ elliptic whenever that is possible. One of the many benefits of such a

formulation is that the discretization and solution processes can be designed for each variable individually, almost independently of the others. This ellipticity suggests that the system is essentially a set of uncoupled elliptic equations in each scalar variable. This is true to some qualitative degree, but this view should not be carried too far. For example, this equivalence implies immediately that an optimal multigrid solver can be designed simply as a diagonal preconditioner that uses multigrid individually on each variable. However, multigrid can generally do much better by applying it directly to the least-squares system: relaxation would be allowed to interact with all variables on all levels, not just on the finest as diagonal preconditioners would do.

Unfortunately, no theory seems to exist that would allow us to claim in any generality that such a fully integrated multigrid scheme is optimal. We are thus compelled to establish this claim theoretically, which we do in the last section by applying the theory developed in [5]. (The only nontrivial task here is the verification of their second smoothing assumption.)

The next section introduces notation, describes the variational approach, and establishes an ellipticity estimate in the necessary form. Approximation by the finite element method and its error analysis are developed in section 3.

2. First-order system least squares. Assume that Ω is a bounded, open, connected domain in $\Re^n$ ($n = 2$ or 3) with Lipschitz boundary $\partial\Omega$. Consider the following second-order elliptic boundary value problem:

$$\left\{\begin{array}{rcll} -\nabla\cdot(A\nabla p) + Xp &=& f, & \text{in} \quad \Omega, \\ p &=& 0, & \text{on} \quad \Gamma_D, \\ \mathbf{n}\cdot A\nabla p &=& 0, & \text{on} \quad \Gamma_N, \end{array}\right. \tag{2.1}$$

where the symbols $\nabla\cdot$ and ∇ stand for the divergence and gradient operators, respectively, A is an $n \times n$ symmetric matrix of functions in $L^\infty(\Omega)$, X is an at most first-order linear differential operator, $\Gamma_D \cup \Gamma_N = \Gamma$ is a partitioning of the boundary of Ω, and $\mathbf{n}$ is the outward unit vector normal to the boundary. We assume that A is uniformly symmetric positive definite and scaled appropriately: there exist positive constants

$$0 < \lambda \le 1 \le \Lambda \tag{2.2}$$

such that

$$\lambda\boldsymbol{\xi}^T\boldsymbol{\xi} \le \boldsymbol{\xi}^T A\boldsymbol{\xi} \le \Lambda\boldsymbol{\xi}^T\boldsymbol{\xi} \tag{2.3}$$

for all $\boldsymbol{\xi} \in \Re^n$ and almost all $x \in \overline{\Omega}$.

Introducing the flux variable

$$\mathbf{u} = A\nabla p,$$

problem (2.1) may be rewritten as a first-order system of partial differential equations as follows:

$$\left\{\begin{array}{rcll} \mathbf{u} - A\nabla p &=& \mathbf{0}, & \text{in} \quad \Omega, \\ \nabla^*\mathbf{u} + Xp &=& f, & \text{in} \quad \Omega, \\ p &=& 0, & \text{on} \quad \Gamma_D, \\ \mathbf{n}\cdot\mathbf{u} &=& 0, & \text{on} \quad \Gamma_N, \end{array}\right. \tag{2.4}$$

where $\nabla^* : H^1(\Omega)^n \to L^2(\Omega)$ is the formal adjoint of $\nabla : H^1(\Omega) \to L^2(\Omega)^n$; that is, $\nabla^* \equiv -\nabla\cdot$. Under appropriate assumptions on Γ_D and X, the associated weak form of the system (2.1) is uniquely solvable in $H^1(\Omega)$ for any $f \in H^{-1}(\Omega)$ or uniquely solvable in $H^1(\Omega)/\Re$ (cf. [20]) if and only if f satisfies the compatibility condition $\int_\Omega f = 0$.

Let $\mathbf{curl} \equiv \nabla\times$ denote the curl operator. (Here and henceforth, we use notation for the case $n = 3$ and consider the special case $n = 2$ in the natural way by identifying $\Re^2$ with the (x_1, x_2)-plane in $\Re^3$. Thus, if $\mathbf{u}$ is two dimensional, then $\nabla\times \mathbf{u} = \mathbf{0}$ means $\partial_1 u_2 - \partial_2 u_1 = 0$, where u_1 and u_2 are the components of $\mathbf{u}$. In section 2.1, we consider only the case $n = 2$, so there we will interpret $\nabla \times \mathbf{u}$ to mean $\partial_1 u_2 - \partial_2 u_1$.) Note that if $\mathbf{u}$ is sufficiently smooth, then the properly scaled solution $A^{-1}\mathbf{u}$ of (2.4) is curl free, i.e., $\nabla\times (A^{-1}\mathbf{u}) = \mathbf{0}$, and the homogeneous Dirichlet boundary condition on Γ_D implies the tangential flux or no-slip condition

$$\boldsymbol{\gamma}_{\boldsymbol{\tau}}(A^{-1}\mathbf{u}) = \mathbf{0}, \quad \text{on } \Gamma_D,$$

where $\boldsymbol{\gamma}_{\boldsymbol{\tau}}\mathbf{u} \equiv \mathbf{n} \times \mathbf{u}$. Here, $\boldsymbol{\tau}$ represents the unit vector tangent to the boundary Γ.

An equivalent extended system for (2.4) is

$$\left\{\begin{array}{rcll} \mathbf{u} - A\nabla p &=& \mathbf{0}, & \text{in } \Omega, \\ \nabla^*\mathbf{u} + Xp &=& f, & \text{in } \Omega, \\ \nabla\times A^{-1}\mathbf{u} &=& \mathbf{0}, & \text{in } \Omega, \\ p &=& 0, & \text{on } \Gamma_D, \\ \mathbf{n}\cdot\mathbf{u} &=& 0, & \text{on } \Gamma_N, \\ \boldsymbol{\gamma}_{\boldsymbol{\tau}}(A^{-1}\mathbf{u}) &=& \mathbf{0}, & \text{on } \Gamma_D. \end{array}\right. \tag{2.5}$$

It is this system that we intend to solve by a least-squares Rayleigh–Ritz discretization and a fully variational multigrid solver.

First we establish notation. Let $\mathcal{D}(\Omega)$ be the linear space of infinitely differentiable functions with compact support on Ω, and

$$\mathcal{D}(\bar{\Omega}) = \{\varphi|_\Omega \,:\, \varphi \in \mathcal{D}(\mathcal{O}) \text{ for some open subset } \Omega \subset \mathcal{O} \subset \Re^n\}.$$

Let $(\cdot\,,\,\cdot)_{0,\Omega}$ denote the inner product on $L^2(\Omega)^n$, $\|\cdot\|_{0,\Omega}$ denote its induced norm, and, for $m \geq 0$, $H^m(\Omega)^n$ denote the standard Sobolev space with norm $\|\cdot\|_{m,\Omega}$ and seminorms $|\cdot|_{i,\Omega}$ $(0 \leq i \leq m)$. (We suppress the subscript n because dependence of the vector norms on dimension will be clear by context.) Let $H^{s-\frac{1}{2}}(\Gamma)$ for $s = 1$ or 2 denote the trace Sobolev space with norm

$$\|r\|_{s-\frac{1}{2},\Gamma} = \inf\{\|v\|_{s,\Omega} : v \in H^s(\Omega),\ \operatorname{trace} v = r \text{ on } \Gamma\}$$

and $H^{-\frac{1}{2}}(\Gamma)$ the dual space for $H^{\frac{1}{2}}(\Gamma)$ with the obvious dual norm

$$\|r^*\|_{-\frac{1}{2},\Gamma} = \sup\left\{\frac{\langle r^*,\, r\rangle}{\|r\|_{1/2,\Gamma}} \,:\, 0 \neq r \in H^{\frac{1}{2}}(\Gamma)\right\},$$

where $\langle\cdot\,,\,\cdot\rangle$ denotes the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$. We use the following spaces to define a least-squares variational form for the extended system (2.5). Let

$$H(\mathrm{div};\,\Omega) = \{\mathbf{v} \in L^2(\Omega)^n \,:\, \nabla^*\mathbf{v} \in L^2(\Omega)\},$$

$$H(\mathbf{curl}\, A;\, \Omega) = \{\mathbf{v} \in L^2(\Omega)^n \,:\, \nabla\times (A^{-1}\mathbf{v}) \in L^2(\Omega)^{2n-3}\}, \tag{2.6}$$

which are Hilbert spaces under the respective norms

$$\|\mathbf{v}\|_{H(\mathrm{div};\,\Omega)} \equiv \left(\|\mathbf{v}\|_{0,\,\Omega}^2 + \|\nabla^*\mathbf{v}\|_{0,\,\Omega}^2\right)^{\frac{1}{2}},$$

$$\|\mathbf{v}\|_{H(\mathbf{curl}\, A;\,\Omega)} \equiv \left(\|\mathbf{v}\|_{0,\,\Omega}^2 + \|\nabla\times (A^{-1}\mathbf{v})\|_{0,\,\Omega}^2\right)^{\frac{1}{2}}.$$

When A is the identity matrix in (2.6), we use the simpler notation $H(\mathbf{curl};\Omega)$. Define the subspaces

$$\mathbf{W}_0(\mathrm{div};\,\Omega) = \{\mathbf{v} \in H(\mathrm{div}\,;\,\Omega) \,:\, \mathbf{n}\cdot\mathbf{v} = \mathbf{0} \text{ on } \Gamma_N\},$$

$$\mathbf{W}_0(\mathbf{curl}\, A;\,\Omega) = \{\mathbf{v} \in H(\mathbf{curl}\, A;\,\Omega) \,:\, \boldsymbol{\gamma}_{\boldsymbol{\tau}}(A^{-1}\mathbf{v}) = \mathbf{0} \text{ on } \Gamma_D\},$$

and

$$\mathbf{W} = \mathbf{W}_0(\mathrm{div}\,;\,\Omega) \cap \mathbf{W}_0(\mathbf{curl}\, A;\,\Omega). \tag{2.7}$$

Finally, define the subspace

$$V = \{q \in H^1(\Omega) :\; q = 0 \text{ on } \Gamma_D\}. \tag{2.8}$$

In [11] the following quadratic functional associated with system (2.4) was examined:

$$G_0(\mathbf{v},\, q;\, f) = \|\mathbf{v} - A\nabla q\|_{0,\,\Omega}^2 + \|\nabla^*\mathbf{v} + Xq - f\|_{0,\,\Omega}^2 \tag{2.9}$$

for $(\mathbf{v},\, q) \in \mathbf{W}_0(\mathrm{div}\,;\,\Omega)\times V$. There it was shown that $G_0(\mathbf{v},\, q;\, 0)$ is equivalent to the $H(\mathrm{div};\,\Omega)\times H^1(\Omega)$ norm on $\mathbf{W}_0(\mathrm{div}\,;\,\Omega)\times V$ under the following *original* assumption.

*Assumption A*0. Either $\Gamma_D \neq \emptyset$ or an additional constraint is imposed on V, such as $\int_\Omega p\, dx = 0$, so that a Poincaré–Friedrichs inequality holds: there exists a constant $d > 0$ depending only on the domain Ω and the uniform bounds on A (see (2.3)) such that

$$\|p\|_{0,\,\Omega}^2 \leq d\|A^{\frac{1}{2}}\nabla p\|_{0,\,\Omega}^2 \tag{2.10}$$

for $p \in V$.

If $\Gamma_D \neq \emptyset$ or $Xp \neq 0$ for $p \equiv$ *constant*, we assume that for any $f \in H^{-1}(\Omega)$ the associated weak form of (2.1) is invertible in $H^1(\Omega)$. If $\Gamma_D = \emptyset$ and $Xp = 0$ for $p \equiv$ *constant*, we assume that the associated weak form of (2.1) is invertible in $H^1(\Omega)/\Re$ for every $f \in H^{-1}(\Omega)$ such that $\int_\Omega f = 0$. In either case, we assume

$$\|Xp\|_{0,\,\Omega} \leq \eta\|A^{\frac{1}{2}}\nabla p\|_{0,\,\Omega} \tag{2.11}$$

for some $\eta > 0$ and every $p \in V$ for which $A\nabla p \in \mathbf{W}_0(\mathrm{div}\,;\,\Omega)$.

The modified quadratic functional we study here is given by

$$G(\mathbf{v},\, q;\, f) = \|\mathbf{v} - A\nabla q\|_{0,\,\Omega}^2 + \|\nabla^*\mathbf{v} + Xq - f\|_{0,\,\Omega}^2 + \|\nabla\times (A^{-1}\mathbf{v})\|_{0,\,\Omega}^2 \tag{2.12}$$

for $(\mathbf{v},\, q) \in \mathbf{W}\times V$. Then the least-squares problem for (2.5) is to minimize this quadratic functional over $\mathbf{W}\times V$: find $(\mathbf{u},\, p) \in \mathbf{W}\times V$ such that

$$G(\mathbf{u},\, p;\, f) = \inf_{(\mathbf{v},\, q)\in\mathbf{W}\times V} G(\mathbf{v},\, q;\, f). \tag{2.13}$$

It is easy to see that the variational form for (2.13) is to find $(\mathbf{u}, p) \in \mathbf{W} \times V$ such that

$$\mathcal{F}(\mathbf{u}, p; \mathbf{v}, q) = f(\mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in \mathbf{W} \times V, \tag{2.14}$$

where the bilinear form $\mathcal{F}(\cdot\,;\cdot) : (\mathbf{W} \times V)^2 \longrightarrow \Re$ is defined by

$$\begin{aligned}\mathcal{F}(\mathbf{u}, p; \mathbf{v}, q) &= (\mathbf{u} - A\nabla p, \mathbf{v} - A\nabla q)_{0,\Omega} + (\nabla^*\mathbf{u} + Xp, \nabla^*\mathbf{v} + Xq)_{0,\Omega} \\ &\quad + (\nabla\times (A^{-1}\mathbf{u}), \nabla\times (A^{-1}\mathbf{v}))_{0,\Omega}\end{aligned} \tag{2.15}$$

and the linear functional $f(\cdot, \cdot) : \mathbf{W} \times V \longrightarrow \Re$ is defined by

$$f(\mathbf{v}, q) = (f, \nabla^*\mathbf{v} + Xq)_{0,\Omega}. \tag{2.16}$$

The first theorem establishes ellipticity and continuity of the bilinear form (2.15) with respect to the $(H(\mathrm{div};\,\Omega)\cap H(\mathbf{curl}\,A;\,\Omega))\times H^1(\Omega)$ norm under only Assumption A0.

THEOREM 2.1. *Assume* A0. *Then there exist positive constants* α_0 *and* α_1 *such that*

$$\alpha_0 \left(\|\mathbf{v}\|^2_{0,\Omega} + \|\nabla^*\mathbf{v}\|^2_{0,\Omega} + \|\nabla\times (A^{-1}\mathbf{v})\|^2_{0,\Omega} + \|q\|^2_{1,\Omega}\right) \le \mathcal{F}(\mathbf{v}, q; \mathbf{v}, q) \tag{2.17}$$

for any $(\mathbf{v}, q) \in \mathbf{W} \times V$ *and*

$$\begin{aligned}\mathcal{F}(\mathbf{u}, p; \mathbf{v}, q) \le \alpha_1 &\left(\|\mathbf{u}\|^2_{0,\Omega} + \|\nabla^*\mathbf{u}\|^2_{0,\Omega} + \|\nabla\times (A^{-1}\mathbf{u})\|^2_{0,\Omega} + \|p\|^2_{1,\Omega}\right)^{\frac{1}{2}} \\ &\cdot \left(\|\mathbf{v}\|^2_{0,\Omega} + \|\nabla^*\mathbf{v}\|^2_{0,\Omega} + \|\nabla\times (A^{-1}\mathbf{v})\|^2_{0,\Omega} + \|q\|^2_{1,\Omega}\right)^{\frac{1}{2}}\end{aligned} \tag{2.18}$$

for any $(\mathbf{u}, p),\ (\mathbf{v}, q) \in \mathbf{W} \times V$.

Proof. This result follows from Theorem 3.1 in [11], where it was shown that the functional

$$G_0(\mathbf{v}, q; f) = \|\mathbf{v} - A\nabla q\|^2_{0,\Omega} + \|\nabla^*\mathbf{v} + Xq - f\|^2_{0,\Omega}$$

and associated bilinear form

$$\mathcal{F}_0(\mathbf{u}, p; \mathbf{v}, q) = (\mathbf{u} - A\nabla p, \mathbf{v} - A\nabla q)_{0,\Omega} + (\nabla^*\mathbf{u} + Xp, \nabla^*\mathbf{v} + Xq)_{0,\Omega}$$

satisfy the bounds (2.17) and (2.18) with the terms involving $\nabla\times$ removed, for all $(\mathbf{u}, p),\ (\mathbf{v}, q) \in \mathbf{W}_0(\mathrm{div}\,;\,\Omega) \times V$. Since $\mathbf{W} \subset \mathbf{W}_0(\mathrm{div}\,;\,\Omega)$, (2.17) and (2.18) hold for $\mathcal{F}$ restricted to $\mathbf{W} \times V$. ∎

The purpose of the remainder of this section is to show that, under some additional hypotheses on A and Ω, the functional (2.12) with $f = 0$ is equivalent to the $H^1(\Omega)^{n+1}$ norm on $\mathbf{W}\times V$. This result requires a generalization of results in [20], which assumed that $A = I,\ X = 0$, and either $\Gamma = \Gamma_D$ or $\Gamma = \Gamma_N$.

In our proof we make use of a decomposition of $\mathbf{W}$ into certain gradient and curl components. We do this by showing that these components satisfy certain boundary value problems, which we ensure are $H^2(\Omega)$ regular by making the following *additional* assumptions.

*Assumption A*1. The domain Ω is bounded, open, and connected in $\Re^n$ ($n = 2$ or 3) with boundary Γ, which consists of a finite number of disjoint, simple, closed

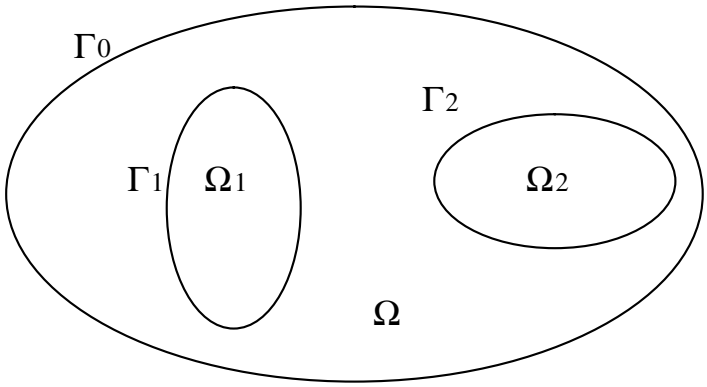

FIG. 1.

curves (surfaces) Γ_i, $i = 0, \ldots, L$; Γ_0 is the outer boundary and Γ_i, $i = 1, \ldots, L$, are $C^{1,1}$ boundaries of a finite number of disjoint holes in Ω (see Fig. 1). Let Ω_i be the interior of Γ_i, $i = 1, \ldots, L$. For $n = 2$, Γ_0 is piecewise $C^{1,1}$ with no reentrant corners, while for $n = 3$, Γ_0 is $C^{1,1}$ or a convex polyhedron.

*Assumption A*2. The boundary is divided into Dirichlet and Neumann parts: $\Gamma = \Gamma_D \cup \Gamma_N$ such that $\Gamma_i \subseteq \Gamma_D$ for $i \in D$ and $\Gamma_i \subseteq \Gamma_N$ for $i \in N$ with $D \cup N = \{1, \ldots, L\}$. For $n = 2$, Γ_0 is divided into a finite number of connected pieces: $\Gamma_0 = \cup_{i=1,\ldots,M} \Gamma_{0,i}$ such that $\Gamma_{0,i} \subseteq \Gamma_D$ for $i \in D_0$ and $\Gamma_{0,i} \subseteq \Gamma_N$ for $i \in N_0$; since Γ_0 is a simple closed curve, M is even; let D_0 be the odd indices and N_0 be the even indices. For $n = 3$, either $\Gamma_0 \subseteq \Gamma_D$ or $\Gamma_0 \subseteq \Gamma_N$.

*Assumption A*3. The matrix A is $C^{1,1}$. If $n = 2$ and $\mathbf{x} \in \Gamma_0$ is a point that separates Γ_D and Γ_N, then $\mathbf{x}$ must be a corner of Γ_0 and $\mathbf{n}_-^T A \mathbf{n}_+ \leq 0$, where $\mathbf{n}_-$ and $\mathbf{n}_+$ are the outward unit normal vectors on the adjacent edges at $\underline{\mathbf{x}}$.

In what follows, we will appeal often to a boundary value problem of the form

$$
\left\{
\begin{array}{rcccll}
\nabla^*(A\nabla p) & = & f, & \text{in} & \Omega, & \\
p & = & g_i, & \text{on} & \Gamma_i, & \text{for } i \in D, \\
p & = & g_{0i}, & \text{on} & \Gamma_{0i}, & \text{for } i \in D_0, \\
\mathbf{n} \cdot A\nabla p & = & h_i, & \text{on} & \Gamma_i, & \text{for } i \in N, \\
\mathbf{n} \cdot A\nabla p & = & h_{0i}, & \text{on} & \Gamma_{0i}, & \text{for } i \in N_0,
\end{array}
\right. \tag{2.19}
$$

where g_i, $g_{0j} \in H^{\frac{3}{2}}(\Gamma_i)$ for $i \in D$, $j \in D_0$, and h_i, $h_{0j} \in H^{\frac{1}{2}}(\Gamma_i)$ for $i \in N$, $j \in N_0$. For our needs, we assume that g_i is nonzero only when Γ_i is $C^{1,1}$. Now our additional assumptions, together with the original ones, are sufficient to guarantee that (2.19) is $H^2(\Omega)$ regular: there exists a constant C depending only on A, X, and Ω such that

$$
\begin{aligned}
\|p\|_{2,\Omega} \leq C \Bigg(& \|f\|_{0,\Omega} + \sum_{i \in D} \|g_i\|_{3/2,\Gamma_i} + \sum_{i \in D_0} \|g_{0i}\|_{3/2,\Gamma_i} + \sum_{i \in N} \|h_i\|_{1/2,\Gamma_i} \\
& + \sum_{i \in N_0} \|h_{0i}\|_{1/2,\Gamma_i} \Bigg).
\end{aligned} \tag{2.20}
$$

This result follows from standard partition of unity arguments (cf. [19]). For both $n = 2$ and $n = 3$, the solution is clearly in H^2 in the interior and along smooth

portions of the boundary. For $n = 2$ the arguments in Chapter 4 of Grisvard [21] can be used near the corners. There, polygonal domains are studied with $A = I$ and H^2 regularity results if there are no reentrant corners and each corner that separates Γ_D from Γ_N has an angle less than $\pi/2$. In our context, we consider $A \in C^{1,1}$. Since A is smooth, there exists a smooth transformation to a problem in a new coordinate system with A replaced by I. The results of Grisvard can be applied in this frame and the inverse transformation yields the criterion in Assumption $A3$. For $n = 3$, Γ_0 is either $C^{1,1}$, which poses no problem or Γ_0 is a convex polyhedron. Since by assumption $\Gamma_0 \subseteq \Gamma_D$ or $\Gamma_0 \subseteq \Gamma_N$ the results in Chapter 8 of [21] imply that a convex polyhedron is sufficient.

We remark that the results in this paper are applicable to any domains for which problems of the type (2.19) are $H^2(\Omega)$ regular. Our assumptions reflect the limit of current knowledge in this respect.

Our main theorem establishes equivalence of the bilinear form (2.15) and the $H^1(\Omega)^{n+1}$ norm under the additional assumptions $A1$–$A3$.

THEOREM 2.2. *Assume A0–A3. Then there exist positive constants α_2 and α_3 such that*

$$\alpha_2 \left(\|\mathbf{v}\|^2_{1,\Omega} + \|q\|^2_{1,\Omega} \right) \leq \mathcal{F}(\mathbf{v}, q; \mathbf{v}, q) \tag{2.21}$$

for any $(\mathbf{v}, q) \in \mathbf{W} \times V$ *and*

$$\mathcal{F}(\mathbf{u}, p; \mathbf{v}, q) \leq \alpha_3 \left(\|\mathbf{u}\|^2_{1,\Omega} + \|p\|^2_{1,\Omega} \right)^{\frac{1}{2}} \left(\|\mathbf{v}\|^2_{1,\Omega} + \|q\|^2_{1,\Omega} \right)^{\frac{1}{2}} \tag{2.22}$$

for any $(\mathbf{u}, p),\ (\mathbf{v}, q) \in \mathbf{W} \times V$.

Proof. In light of Theorem 2.1, to prove Theorem 2.2 we need only show that $\mathbf{W}$ is algebraically and topologically included in $H^1(\Omega)^n$; that is, for any $\mathbf{v} \in \mathbf{W}$ there exist constants $\hat{\alpha}_2$ and $\hat{\alpha}_3$ such that

$$\hat{\alpha}_2 \|\mathbf{v}\|^2_{1,\Omega} \leq \|\mathbf{v}\|^2_{0,\Omega} + \|\nabla^* \mathbf{v}\|^2_{0,\Omega} + \|\nabla \times A^{-1} \mathbf{v}\|^2_{0,\Omega} \leq \hat{\alpha}_3 \|\mathbf{v}\|^2_{1,\Omega}. \tag{2.23}$$

The upper bound in (2.23) follows from the triangle inequality. We prove the lower bound for $n = 2$ and 3 separately. The two-dimensional result could have been deduced as a special case of the three-dimensional result, but it would then have inherited the more restrictive assumptions.

2.1. Two dimensions. In this section, we interpret the curl of a vector function $\mathbf{u}$ to mean the scalar function $\nabla \times \mathbf{u} = \partial_1 u_2 - \partial_2 u_1$. Note that, for $n = 2$, the operator $\nabla^\perp$ defined by

$$\nabla^\perp q \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla q = \begin{pmatrix} \partial_2 q \\ -\partial_1 q \end{pmatrix}$$

is the formal adjoint of $\nabla\times$:

$$\nabla \times \mathbf{v} = \nabla^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{v}.$$

Let

$$P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

then

$$P\nabla = \nabla^{\perp}, \quad P^*\nabla^{\perp} = \nabla,$$
$$\nabla^* P^* = \nabla\times, \quad \nabla\times P = \nabla^*. \tag{2.24}$$

Let $\mathbf{n} = (n_1,\ n_2)^t$ be the outward unit normal and let $\boldsymbol{\tau} = (\tau_1,\ \tau_2)^t$ be the unit tangent oriented clockwise on Γ_0. Then $\boldsymbol{\tau} = P\mathbf{n}$. Many general results involving ∇^* and ∇ can be restated for $\nabla\times$ and $\nabla^{\perp}$ by using P. In [20] we find the following result.

LEMMA 2.1. *Let $\Omega \subset \Re^2$; then $\mathbf{w} \in H(\mathrm{div};\,\Omega)$ such that $\nabla^*\mathbf{w} = 0$ and $\int_{\Gamma_i} \mathbf{n}\cdot\mathbf{w} = 0$ for $i = 1,\ldots,L$ if and only if $\mathbf{w} = \nabla^{\perp} q$ with $q \in H^1(\Omega)$.*

Proof. See Theorem 3.1 in Chapter I of [20]. ∎

This becomes the following.

LEMMA 2.2. *Let $\Omega \subset \Re^2$; then $\mathbf{w} \in H(\mathbf{curl}\,;\Omega)$ such that $\nabla\times\mathbf{w} = 0$ and $\int_{\Gamma_i} \boldsymbol{\tau}\cdot\mathbf{w} = 0$ for $i = 1,\ldots,L$ if and only if $\mathbf{w} = \nabla q$ with $q \in H^1(\Omega)$.*

Proof. The proof follows from Lemma 2.1 and (2.24). ∎

A result analogous to Green's formula also follows:

$$(\nabla\times\mathbf{z},\ \phi) = (\mathbf{z},\ \nabla^{\perp}\phi) - \int_{\Gamma} (\boldsymbol{\tau}\cdot\mathbf{z})\phi \tag{2.25}$$

for $\mathbf{z} \in H(\mathbf{curl}\,;\Omega)$ and $\phi \in H^1(\Omega)$.

The next lemma obtains sufficient conditions for a vector function in $\mathbf{W}$ to be zero.

LEMMA 2.3. *Let A be uniformly symmetric positive definite on Ω, which satisfies Assumptions A1 and A2. Let $\mathbf{z} \in \mathbf{W}$ satisfy*

$$\begin{array}{lrcll}
\text{i)} & \nabla^*\mathbf{z} & = & 0, & \text{in } \Omega,\\
\text{ii)} & \nabla\times A^{-1}\mathbf{z} & = & 0, & \text{in } \Omega,\\
\text{iii)} & \int_{\Gamma_i} \mathbf{n}\cdot\mathbf{z} & = & 0, & \text{for } i \in D,\\
\text{iv)} & \int_{\Gamma_i} \boldsymbol{\tau}\cdot A^{-1}\mathbf{z} & = & 0, & \text{for } i \in N,\\
\text{and either} & & & & \\
\text{v)} & \int_{\Gamma_{0j}} \mathbf{n}\cdot\mathbf{z} & = & 0, & \text{for } j \in D_0,\\
\text{or} & & & & \\
\text{vi)} & \int_{\Gamma_{0j}} \boldsymbol{\tau}\cdot A^{-1}\mathbf{z} & = & 0, & \text{for } j \in N_0.
\end{array} \tag{2.26}$$

Then $\mathbf{z} = \mathbf{0}$.

Proof. Assumptions (2.26) i), ii), iii), and iv) together with Lemmas 2.1 and 2.2 yield

$$\mathbf{z} = A\nabla p, \qquad \mathbf{z} = \nabla^{\perp}\phi,$$

with $p,\ \phi \in H^1(\Omega)$. Using Green's formula and assumption i), we have

$$\begin{aligned}(\nabla^*\mathbf{z},\ p) &= (\mathbf{z},\ \nabla p) - \int_{\Gamma} (\mathbf{n}\cdot\mathbf{z})p\\ &= (A^{-1}\mathbf{z},\ \mathbf{z}) - \int_{\Gamma} (\mathbf{n}\cdot\mathbf{z})p = 0.\end{aligned}$$

Thus,

$$(A^{-1}\mathbf{z},\ \mathbf{z}) = \int_{\Gamma_0} (\mathbf{n}\cdot\mathbf{z})p + \sum_{i\in D}\int_{\Gamma_i} (\mathbf{n}\cdot\mathbf{z})p + \sum_{i\in N}\int_{\Gamma_i} (\mathbf{n}\cdot\mathbf{z})p. \tag{2.27}$$

The last sum is zero because $\mathbf{n}\cdot\mathbf{z}=0$ on Γ_N. The second sum is also zero because integration by parts on each of its terms yields

$$\int_{\Gamma_i}(\mathbf{n}\cdot\mathbf{z})p=-\int_{\Gamma_i}(\boldsymbol{\tau}\cdot\nabla\phi)p=\int_{\Gamma_i}(\boldsymbol{\tau}\cdot\nabla p)\phi=\int_{\Gamma_i}(\boldsymbol{\tau}\cdot A^{-1}\mathbf{z})\phi=0,$$

since the integration is around a closed path and $\boldsymbol{\tau}\cdot A^{-1}\mathbf{z}=0$ on Γ_D.

To prove that the first term on the right-hand side of (2.27) is also zero, assume first that (2.26) v) holds. Since $\mathbf{z}\in\mathbf{W}$, then $\mathbf{n}\cdot\mathbf{z}=\mathbf{0}$ on Γ_N and we have

$$\begin{aligned}\int_{\Gamma_0}(\mathbf{n}\cdot\mathbf{z})p&=\sum_{j\in D_0}\int_{\Gamma_{0j}}(\mathbf{n}\cdot\mathbf{z})p+\sum_{j\in N_0}\int_{\Gamma_{0j}}(\mathbf{n}\cdot\mathbf{z})p\\&=\sum_{j\in D_0}\int_{\Gamma_{0j}}(\mathbf{n}\cdot\mathbf{z})p.\end{aligned}$$

Using this relation and noting that $\boldsymbol{\tau}\cdot A^{-1}\mathbf{z}=\boldsymbol{\tau}\cdot\nabla p=0$ on Γ_D, which implies that $p=\alpha_j$ on Γ_{0j} for $j\in D_0$ and some constant α_j, we have

$$\int_{\Gamma_0}(\mathbf{n}\cdot\mathbf{z})p=\sum_{j\in D_0}\alpha_j\int_{\Gamma_{0j}}\mathbf{n}\cdot\mathbf{z}=0$$

by assumption v).

Next, assume that case (2.26) vi) holds. Since the integration is over a closed path, and $\boldsymbol{\tau}\cdot\nabla p=\boldsymbol{\tau}\cdot A^{-1}\mathbf{z}=0$ on Γ_D, we have

$$\begin{aligned}\int_{\Gamma_0}(\mathbf{n}\cdot\mathbf{z})p&=-\int_{\Gamma_0}(\boldsymbol{\tau}\cdot\nabla\phi)p\\&=\int_{\Gamma_0}(\boldsymbol{\tau}\cdot\nabla p)\phi\\&=\sum_{j\in D_0}\int_{\Gamma_{0j}}(\boldsymbol{\tau}\cdot\nabla p)\phi+\sum_{j\in N_0}\int_{\Gamma_{0j}}(\boldsymbol{\tau}\cdot\nabla p)\phi\\&=\sum_{j\in N_0}\int_{\Gamma_{0j}}(\boldsymbol{\tau}\cdot\nabla p)\phi.\end{aligned}$$

Similar to the above, using this relation and noting that $\mathbf{n}\cdot\mathbf{z}=-\boldsymbol{\tau}\cdot\nabla\phi=0$ on Γ_N, which implies that $\phi=\beta_j$ on Γ_{0j} for $j\in N_0$ and some constant β_j, we have

$$\int_{\Gamma_0}(\mathbf{n}\cdot\mathbf{z})p=\sum_{j\in N_0}\beta_j\int_{\Gamma_{0j}}\boldsymbol{\tau}\cdot A^{-1}\mathbf{z}=0$$

by assumption vi).

In either case, we thus have

$$(A^{-1}\mathbf{z},\ \mathbf{z})=0.$$

Since A is uniformly symmetric positive definite, it follows that $\mathbf{z}=\mathbf{0}$. □

We now construct a basis for the functions in $\mathbf{W}$ that satisfy (2.26) i) and ii). Consider the functions p_i for $i\in D$ that satisfy

$$\left\{\begin{array}{rcll}\nabla^*A\nabla p_i&=&0,&\text{in}\quad\Omega,\\p_i&=&1,&\text{on}\quad\Gamma_i,\\p_i&=&0,&\text{on}\quad\Gamma_D\setminus\Gamma_i,\\\mathbf{n}\cdot A\nabla p_i&=&0,&\text{on}\quad\Gamma_N,\end{array}\right.\tag{2.28}$$

and the functions p_{0j} for $j \in D_0$ that satisfy

$$\text{(2.29)} \qquad \begin{cases} \nabla^* A \nabla p_{0j} &= 0, & \text{in} & \Omega, \\ p_{0j} &= 1, & \text{on} & \Gamma_{0j}, \\ p_{0j} &= 0, & \text{on} & \Gamma_D \setminus \Gamma_{0j}, \\ \mathbf{n} \cdot A \nabla p_{0j} &= 0, & \text{on} & \Gamma_N. \end{cases}$$

Clearly, $A\nabla p_i$, $A\nabla p_{0j} \in \mathbf{W}$, and they satisfy (2.26) i) and ii). Suppose that Assumptions $A1$–$A3$ hold. Then we can recast (2.29) so that it has homogeneous boundary data but a nonzero source term bounded in $L^2(\Omega)$ by a constant depending only on Ω. (This is easily done by extending the special Dirichlet data of (2.29) smoothly into Ω while satisfying the homogeneous Neumann boundary conditions, then restating (2.29) as an equation for the difference.) Thus, (2.20) applies so that both (2.28) and (2.29) are $H^2(\Omega)$ regular and

$$\text{(2.30)} \qquad \|p_i\|_{2,\Omega} \le C_i, \quad \|p_{0j}\|_{2,\Omega} \le C_{0j},$$

for $i \in D$ and $j \in D_0$, where C_i and C_{0j} depend only on Ω and A.

Next, note that

$$\text{(2.31)} \qquad \nabla \times A^{-1} \nabla^\perp = \nabla^* P^* A^{-1} P \nabla = \nabla^* B \nabla,$$

where

$$\text{(2.32)} \qquad B \equiv P^* A^{-1} P = \frac{1}{\det(A)} A.$$

This relation is easily verified algebraically for any 2×2 symmetric matrix A. Now B is uniformly symmetric positive definite:

$$\frac{1}{\Lambda} \boldsymbol{\xi}^T \boldsymbol{\xi} \le \boldsymbol{\xi}^T B \boldsymbol{\xi} \le \frac{1}{\lambda} \boldsymbol{\xi}^T \boldsymbol{\xi}$$

for all $\boldsymbol{\xi} \in R^n$ and $\mathbf{x} \in \bar{\Omega}$. Consider the functions ϕ_i for $i \in N$ that satisfy

$$\text{(2.33)} \qquad \begin{cases} \nabla^* B \nabla \phi_i &= 0, & \text{in} & \Omega, \\ \phi_i &= 1, & \text{on} & \Gamma_i, \\ \phi_i &= 0, & \text{on} & \Gamma_N \setminus \Gamma_i, \\ \mathbf{n} \cdot B \nabla \phi_i &= 0, & \text{on} & \Gamma_D, \end{cases}$$

and the functions ϕ_{0j} for $j \in N_0$ that satisfy

$$\text{(2.34)} \qquad \begin{cases} \nabla^* B \nabla \phi_{0j} &= 0, & \text{in} & \Omega, \\ \phi_{0j} &= 1, & \text{on} & \Gamma_{0j}, \\ \phi_{0j} &= 0, & \text{on} & \Gamma_N \setminus \Gamma_{0j}, \\ \mathbf{n} \cdot B \nabla \phi_{0j} &= 0, & \text{on} & \Gamma_D. \end{cases}$$

Since

$$\begin{aligned} \boldsymbol{\tau} \cdot A^{-1} \nabla^\perp \phi_i &= \mathbf{n} \cdot B \nabla \phi_i = 0, && \text{on } \Gamma_D, \\ \mathbf{n} \cdot \nabla^\perp \phi_i &= \boldsymbol{\tau} \cdot \nabla \phi_i = 0, && \text{on } \Gamma_N, \\ \boldsymbol{\tau} \cdot A^{-1} \nabla^\perp \phi_{0j} &= \mathbf{n} \cdot B \nabla \phi_{0j} = 0, && \text{on } \Gamma_D, \\ \mathbf{n} \cdot \nabla^\perp \phi_{0j} &= \boldsymbol{\tau} \cdot \nabla \phi_{0j} = 0, && \text{on } \Gamma_N, \end{aligned}$$

it follows from (2.31), (2.33), and (2.34) that $\nabla^\perp \phi_i, \nabla^\perp \phi_{0j} \in \mathbf{W}$, and that they satisfy (2.26) i) and ii). We note that if Assumptions $A1$–$A3$ hold, then problems (2.33) and (2.34) satisfy our additional assumptions: the uniformly symmetric positive definite matrix B is $C^{1,1}$ and, if $\underline{x} \in \Gamma_0$ separates Γ_N and Γ_D, then

$$\mathbf{n}_-^T B \mathbf{n}_+ = \frac{1}{\det(A)} \mathbf{n}_-^T A \mathbf{n}_+ \leq 0.$$

Thus, since (2.20) applies as before, then both (2.33) and (2.34) are $H^2(\Omega)$ regular and there exist constants C_i' and C_{0j}' such that

$$\|\phi_i\|_{2,\Omega} \leq C_i', \quad \|\phi_{0j}'\|_{2,\Omega} \leq C_{0j}' \tag{2.35}$$

for $i \in N$ and $j \in N_0$, where C_i' and C_{0j}' depend only on Ω and A.

LEMMA 2.4. *Let A be $C^{1,1}$ and uniformly symmetric positive definite on Ω, which satisfies Assumptions $A1$ and $A2$. Let $\mathbf{z} \in \mathbf{W}$ satisfy*

$$\begin{aligned} &\text{i)} \quad \nabla^* \mathbf{z} &&= 0, \quad \textit{in } \Omega, \\ &\text{ii)} \quad \nabla \times A^{-1} \mathbf{z} &&= 0, \quad \textit{in } \Omega. \end{aligned} \tag{2.36}$$

Then

$$\mathbf{z} = \sum_{j \in D_0} \alpha_{0j} A \nabla p_{0j} + \sum_{i \in D} \alpha_i A \nabla p_i + \sum_{j \in N_0} \beta_{0j} \nabla^\perp \phi_{0j} + \sum_{i \in N} \beta_i \nabla^\perp \phi_i. \tag{2.37}$$

Moreover, if Assumption $A3$ holds, then $\mathbf{z} \in H^1(\Omega)^2$ and

$$\|\mathbf{z}\|_{1,\Omega} \leq C \|\mathbf{z}\|_{0,\Omega}, \tag{2.38}$$

where C depends only on Ω and A.

Proof. We begin by constructing a function $\mathbf{w}$ in one of two ways:

$$\mathbf{w} = \mathbf{z} - \left(\sum_{j \in D_0} \alpha_{0j} A \nabla p_{0j} + \sum_{i \in D} \alpha_i A \nabla p_i + \sum_{i \in N} \beta_i \nabla^\perp \phi_i \right) \tag{2.39}$$

or

$$\mathbf{w} = \mathbf{z} - \left(\sum_{i \in D} \alpha_i A \nabla p_i + \sum_{j \in N_0} \beta_{0j} \nabla^\perp \phi_{0j} + \sum_{i \in N} \beta_i \nabla^\perp \phi_i \right). \tag{2.40}$$

We will then show that proper choices of coefficients lead to the conclusion that $\mathbf{w} = \mathbf{0}$. To this end, choose α_{0j}, α_i, β_{0j}, and β_i so that

$$\int_{\Gamma_i} \mathbf{n} \cdot \mathbf{w} = 0, \quad i \in D \ , \quad \int_{\Gamma_i} \boldsymbol{\tau} \cdot A^{-1} \mathbf{w} = 0, \quad i \in N, \tag{2.41}$$

and

$$\text{either} \quad \int_{\Gamma_{0j}} \mathbf{n} \cdot \mathbf{w} = 0, \quad j \in D_0, \quad \text{or} \quad \int_{\Gamma_{0j}} \boldsymbol{\tau} \cdot A^{-1} \mathbf{w} = 0, \quad j \in N_0. \tag{2.42}$$

A decomposition of the form (2.39) is accomplished by solving the linear system

(2.43)
$$\begin{array}{llll}
\sum_{j\in D_0}\left(\int_{\Gamma_{0i}}\mathbf{n}\cdot A\nabla p_{0j}\right)\alpha_{0j} & +\sum_{j\in D}\left(\int_{\Gamma_{0i}}\mathbf{n}\cdot A\nabla p_j\right)\alpha_j & +\sum_{j\in N}\left(\int_{\Gamma_{0i}}\mathbf{n}\cdot\nabla^{\perp}\phi_j\right)\beta_j & =\int_{\Gamma_{0i}}\mathbf{n}\cdot\mathbf{z} \quad \text{for } i\in D_0,\\
\sum_{j\in D_0}\left(\int_{\Gamma_i}\mathbf{n}\cdot A\nabla p_{0j}\right)\alpha_{0j} & +\sum_{j\in D}\left(\int_{\Gamma_i}\mathbf{n}\cdot A\nabla p_j\right)\alpha_j & & =\int_{\Gamma_i}\mathbf{n}\cdot\mathbf{z} \quad \text{for } i\in D,\\
\sum_{j\in D_0}\left(\int_{\Gamma_i}\boldsymbol{\tau}\cdot\nabla p_{0j}\right)\alpha_{0j} & & +\sum_{j\in N}\left(\int_{\Gamma_i}\mathbf{n}\cdot B\nabla\phi_j\right)\beta_j & =\int_{\Gamma_i}\boldsymbol{\tau}\cdot A^{-1}\mathbf{z} \quad \text{for } i\in N.
\end{array}$$

Note that $\int_{\Gamma_i}\mathbf{n}\cdot\nabla^{\perp}\phi_j=\int_{\Gamma_i}\boldsymbol{\tau}\cdot\nabla\phi_j=0$ for $i\in D$ and $\int_{\Gamma_i}\boldsymbol{\tau}\cdot\nabla p_j=0$ for $i\in N$ because the integrations are carried out on a closed path.

To see that (2.43) has a solution, note first that it is a singular but consistent system of linear equations. Consider the first two block rows in the upper left of the tableau. Since each $A\nabla p_i$, $A\nabla p_{0j}$, and $\nabla^{\perp}\phi_j$ is divergence free, then $\int_{\Gamma}\mathbf{n}\cdot A\nabla p_i=\int_{\Gamma}\mathbf{n}\cdot A\nabla p_{0i}=\int_{\Gamma}\mathbf{n}\cdot\nabla^{\perp}\phi_j 0$. Thus, the sum of any column of these two block rows is zero. The sum of the first two blocks of the right-hand side is also zero by the same reasoning. The null space of the transpose is the same as the null space of the matrix and consists of setting $\alpha_i=\alpha$ for $i\in D$, $\alpha_{0i}=\alpha$ for $i\in D_0$, and $\beta_i=0$ for $i\in N$. This corresponds to a constant function, which is in the null space of ∇. A reduced nonsingular system can be found by setting any α_i or α_{0i} to zero and deleting the corresponding row. To see that this reduced system is nonsingular, assume otherwise; then, for some $\mathbf{z}$, there are two solutions whose α_i's differ by something other than a constant; their difference would yield a nonzero function of the form (2.37) that satisfies the hypotheses of Lemma 2.3, which is a contradiction.

With this choice for α_{0j}, α_i, and β_i, the function $\mathbf{w}$ satisfies the hypotheses of Lemma 2.3, which implies $\mathbf{w}=\mathbf{0}$. If the form (2.40) had been chosen, a similar argument would yield coefficients α_i, β_{0j}, and β_i with one β set to zero.

Now suppose Assumption $A3$ holds. Since the linear system represented by the left-hand side of (2.43) depends only upon Ω and A, then there exist constants C_1 – C_4 such that

$$\begin{aligned}
\max_{j\in D\cup D_0}|\alpha_j|+\max_{j\in N\cup N_0}|\beta_j| &\le C_1\left(\max_{j\in D\cup D_0}\left|\int_{\Gamma_j}\mathbf{n}\cdot\mathbf{z}\right|+\max_{j\in N\cup N_0}\left|\int_{\Gamma_i}\boldsymbol{\tau}\cdot A^{-1}\mathbf{z}\right|\right)\\
&\le C_2(\|\mathbf{n}\cdot\mathbf{z}\|_{-1/2,\,\Gamma}+\|\boldsymbol{\tau}\cdot A^{-1}\mathbf{z}\|_{-1/2,\,\Gamma})\\
&\le C_2(\|\mathbf{z}\|_{H(\mathrm{div};\,\Omega)}+\|A^{-1}\mathbf{z}\|_{H(\mathbf{curl}\,;\,\Omega)})\\
&\le C_3\|\mathbf{z}\|_{0,\,\Omega}.
\end{aligned}$$

Finally, (2.30), (2.35), and (2.37) yield

$$\|\mathbf{z}\|_{1,\,\Omega}\le C_4\left(\max_{i\in D\cup D_0}|\alpha_i|+\max_{i\in N\cup N_0}|\beta_i|\right)\le C\|\mathbf{z}\|_{0,\,\Omega},$$

and the lemma is proved. □

We remark that the decomposition of $\mathbf{z}$ is not unique. For example, any linear combination of (2.39) and (2.40) whose coefficients sum to one again yields zero.

Proof of Theorem 2.2. We now prove the lower bound in (2.23) by decomposing $\mathbf{v}$ as

$$\mathbf{v}=A\nabla p+\nabla^{\perp}\phi, \tag{2.44}$$

where p, $\phi \in H^2(\Omega)$. This is done by first choosing p_0 to satisfy

$$\left\{ \begin{array}{rcll} \nabla^* A \nabla\, p_0 &=& \nabla^* \mathbf{v}, & \text{in} \quad \Omega, \\ p_0 &=& 0, & \text{on} \quad \Gamma_D, \\ \mathbf{n} \cdot A \nabla\, p_0 &=& 0, & \text{on} \quad \Gamma_N, \end{array} \right. \tag{2.45}$$

which by our assumptions is $H^2(\Omega)$ regular and thus (2.20) holds. Together with (2.3) and Assumption $A3$, this implies that there exist constants C_1 and C_2 such that

$$\|A \nabla p_0\|_{1,\,\Omega} \leq C_1 \|p_0\|_{2,\,\Omega} \leq C_2 \|\nabla^* \mathbf{v}\|_{0,\,\Omega}. \tag{2.46}$$

Note also that $p_0 = 0$ on Γ_D, which implies $\boldsymbol{\tau} \cdot A^{-1}(A \nabla p_0) = \boldsymbol{\tau} \cdot \nabla p_0 = 0$ on Γ_D. Together with $\mathbf{n} \cdot (A \nabla p_0) = 0$ on Γ_N, we see that $A \nabla p_0 \in \mathbf{W}$.

Next we construct ϕ_0 to satisfy

$$\left\{ \begin{array}{rcll} \nabla^* B \nabla\, \phi_0 &=& \nabla \times A^{-1} \mathbf{v}, & \text{in} \quad \Omega, \\ \phi_0 &=& 0, & \text{on} \quad \Gamma_N, \\ \mathbf{n} \cdot B \nabla \phi_0 &=& 0, & \text{on} \quad \Gamma_D. \end{array} \right. \tag{2.47}$$

Again we see that (2.47) is $H^2(\Omega)$ regular and there exists a constant C_3 such that

$$\|\nabla^\perp \phi_0\|_{1,\,\Omega} \leq \|\phi_0\|_{2,\,\Omega} \leq C_3 \|\nabla \times A^{-1} \mathbf{v}\|_{0,\,\Omega}. \tag{2.48}$$

Moreover, by similar arguments, we have $\nabla^\perp \phi_0 \in \mathbf{W}$.

Now, let

$$\mathbf{z} = \mathbf{v} - A \nabla p_0 - \nabla^\perp \phi_0. \tag{2.49}$$

Then $\mathbf{z}$ satisfies the hypotheses of Lemma 2.4. Combining (2.49), Lemma 2.4, (2.46), and (2.48) yields

$$\begin{aligned} \|\mathbf{v}\|_{1,\,\Omega} &\leq \|\mathbf{z}\|_{1,\,\Omega} + \|A \nabla p_0\|_{1,\,\Omega} + \|\nabla^\perp \phi_0\|_{1,\,\Omega} \\ &\leq C\, \|\mathbf{z}\|_{0,\,\Omega} + \|A \nabla p_0\|_{1,\,\Omega} + \|\nabla^\perp \phi_0\|_{1,\,\Omega} \\ &\leq C\, (\|\mathbf{v}\|_{0,\,\Omega} + \|A \nabla p_0\|_{0,\,\Omega} + \|\nabla^\perp \phi_0\|_{0,\,\Omega}) + \|A \nabla p_0\|_{1,\,\Omega} + \|\nabla^\perp \phi_0\|_{1,\,\Omega} \\ &\leq C\, \|\mathbf{v}\|_{0,\,\Omega} + C_4 \|A \nabla p_0\|_{1,\,\Omega} + C_5 \|\nabla^\perp \phi_0\|_{1,\,\Omega} \\ &\leq C_6\ \left(\|\mathbf{v}\|_{0,\,\Omega} + \|\nabla^* \mathbf{v}\|_{0,\,\Omega} + \|\nabla \times A^{-1} \mathbf{v}\|_{0,\,\Omega}\right) \end{aligned}$$

for some constant C_6, and the theorem is proved. ∎

Finally, we remark that the full decomposition of $\mathbf{v}$ takes the form

$$\mathbf{v} = A \nabla p_0 + \sum_{j \in D_0} \alpha_{0j} A \nabla p_{0j} + \sum_{i \in D} \alpha_i A \nabla p_i + \nabla^\perp \phi_0 + \sum_{j \in N_0} \beta_{0j} \nabla^\perp \phi_{0j} + \sum_{i \in N} \beta_i \nabla^\perp \phi_i,$$

where α_i and β_i are chosen as in Lemma 2.4.

2.2. Three dimensions. Our additional assumptions for $n = 3$ restrict the boundary Γ_0 to be either Dirichlet or Neumann; that is, $\Gamma_0 \subseteq \Gamma_D$ or $\Gamma_0 \subseteq \Gamma_N$. Further, Γ_0 is now either $C^{1,1}$ or a convex polyhedron. The results in this section generalize Theorems 3.7, 3.8, and 3.9 in Chapter I of [20], where, in addition to the above restrictions on Γ_0, it is assumed that the entire boundary is either Dirichlet or

Neumann and that $A = I$. Unlike the two-dimensional proof, we use the result in [20] in our three-dimensional proof, and thus make the same assumptions on Γ_0.

THEOREM 2.3. *Let $\Omega \in \Re^3$ satisfy Assumption A1. Assume that $A = I$ and that either $\Gamma_N = \Gamma$ or $\Gamma_D = \Gamma$. Then Theorem 2.2 holds.*

Proof. The proof follows from Theorems 3.7 and 3.9 in Chapter I of [20] and Theorem 3.1 in [11]. ∎

Given the more general assumptions on A, Γ_D, and Γ_N, the upper bound in (2.23) is immediate, so our task is again to establish the lower bound in (2.23). We first gather some tools. The next two lemmas are technical but essential to what follows.

LEMMA 2.5. *Let $\Omega \subset \Re^3$ such that the boundary $\partial\Omega$ is piecewise $C^{1,1}$. If $\mathbf{v} \in H(\mathbf{curl}\,;\Omega)$ and $\mathbf{n}\times\mathbf{v} = \mathbf{0}$ on $\hat{\Gamma} \subseteq \Gamma$, then $\mathbf{n}\cdot(\nabla\times\mathbf{v}) = 0$ on $\hat{\Gamma}$ (in the trace sense).*

Proof. The proof follows from a modification to Remark 2.5 in [20]. We offer the following heuristic proof. First, since $\mathbf{v} \in H(\mathbf{curl}\,;\Omega)$, then $\mathbf{n}\times\mathbf{v}$ is well defined on Γ. Also, $\nabla\times\mathbf{v} \in H(\mathrm{div};\Omega)$ so $\mathbf{n}\cdot(\nabla\times\mathbf{v})$ is well defined on Γ. Now $\mathbf{n}\times\mathbf{v} = \mathbf{0}$ on $\hat{\Gamma}$ implies $\mathbf{v}$ is normal to $\hat{\Gamma}$. Assume that $\mathbf{v} \in \mathcal{D}(\overline{\Omega})$ and let $\mathbf{x} \in \hat{\Gamma}$. Consider the definition

$$\nabla\times\mathbf{v}(\mathbf{x}) = \lim_{\Delta\mathcal{V}\to 0} \frac{1}{\Delta\mathcal{V}} \int_{\partial\mathcal{V}} \hat{\mathbf{n}}\times\mathbf{v},$$

where the limit is taken over any convenient neighborhood $\mathcal{V} \subset \Omega$ of $\mathbf{x}$ with volume $\Delta\mathcal{V}$ and surface normal $\hat{\mathbf{n}}$. For example, since Γ is piecewise $C^{1,1}$, we may choose cubes with two sides tangent to the boundary, on which $\hat{\mathbf{n}} = \mathbf{n}$, which yields $\mathbf{n}\cdot(\hat{\mathbf{n}}\times\mathbf{v}) = \mathbf{v}\cdot(\hat{\mathbf{n}}\times\mathbf{n}) = \mathbf{0}$; on the other cube sides, we have the limiting property $\mathbf{n}\cdot(\hat{\mathbf{n}}\times\mathbf{v})\frac{\Delta\mathcal{A}}{\Delta\mathcal{V}} \to 0$ since $\mathbf{v}$ is normal to $\partial\Omega$ at $\mathbf{x}$. The result for $\mathbf{v} \in H(\mathbf{curl}\,;\Omega)$ follows by continuity. ∎

LEMMA 2.6. *Let $\Omega \subset \Re^3$ and let Γ be a simple, closed, piecewise $C^{1,1}$ surface in Ω. Let $p \in H^1(\Omega)$ and $\boldsymbol{\psi} \in H^1(\Omega)^3$. Then*

$$\int_\Gamma ((\mathbf{n}\cdot\nabla\times\boldsymbol{\psi})p + (\mathbf{n}\times\boldsymbol{\psi})\cdot\nabla p) = 0. \tag{2.50}$$

Proof. The proof follows from a modification to Remark 2.5 in [20]. We offer the following heuristic proof. Let $\mathbf{n} = (n_1,\ n_2,\ n_3)$, $\boldsymbol{\psi} = (\psi_1,\ \psi_2,\ \psi_3)$. Then

$$\mathbf{n}\times\boldsymbol{\psi} = \begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad \nabla\times\boldsymbol{\psi} = \begin{pmatrix} 0 & \partial_3 & -\partial_2 \\ -\partial_3 & 0 & \partial_1 \\ \partial_2 & -\partial_1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}.$$

Notice that each row of the matrix defining $\mathbf{n}\times$ is a vector tangent to the surface, say $\boldsymbol{\tau}_1,\ \boldsymbol{\tau}_2,\ \boldsymbol{\tau}_3$. A little algebra on (2.50) yields

$$\int_\Gamma ((\mathbf{n}\cdot\nabla\times\boldsymbol{\psi})p + (\mathbf{n}\times\boldsymbol{\psi})\cdot\nabla p) = \sum_{i=1}^{3} \int_\Gamma \boldsymbol{\tau}_i\cdot\nabla(\psi_i p).$$

Since the surface is piecewise $C^{1,1}$ and closed, each term in the sum is zero. ∎

The next two theorems summarize results found in [20] that we will need.

THEOREM 2.4. *Let $\Omega \subset \Re^3$ satisfy Assumption A1.*

a. *Then*

$$\mathbf{w} \in H(\mathrm{div};\Omega),\quad \nabla^*\mathbf{w} = 0,\quad \int_{\Gamma_i} \mathbf{n}\cdot\mathbf{w} = 0 \ \text{ for } i = 0,\ldots,L \tag{2.51}$$

if and only if there exists $\boldsymbol{\psi} \in H(\mathbf{curl}\,;\Omega)$ *such that*

$$\mathbf{w} = \nabla\times\boldsymbol{\psi}. \tag{2.52}$$

b. *Given* (2.51)*, then there exists* $\boldsymbol{\psi} \in H^1(\Omega)^3$ *such that*

$$\mathbf{w} = \nabla\times\boldsymbol{\psi}, \quad \nabla^*\boldsymbol{\psi} = 0. \tag{2.53}$$

c. *Given* (2.51)*, then there exists* $\boldsymbol{\psi} \in H(\mathbf{curl}\,;\Omega)$ *such that*

$$\mathbf{w} = \nabla\times\boldsymbol{\psi}, \quad \nabla^*\boldsymbol{\psi} = 0, \quad \mathbf{n}\cdot\boldsymbol{\psi} = 0 \quad \textit{on } \Gamma. \tag{2.54}$$

If Ω *is simply connected, then* ψ *is unique. Moreover, if* Γ *is* $C^{1,1}$*, then* $\boldsymbol{\psi} \in H^1(\Omega)^3$*.*

d. *Given* (2.51) *and, in addition,* Ω *is simply connected and* $\mathbf{n}\cdot\mathbf{w} = 0$ *on* Γ*, then there exists a unique* $\boldsymbol{\psi} \in H(\mathbf{curl}\,;\Omega)$ *such that*

$$\mathbf{w} = \nabla\times\boldsymbol{\psi}, \quad \nabla^*\boldsymbol{\psi} = 0, \quad \mathbf{n}\times\boldsymbol{\psi} = \mathbf{0} \ \textit{on } \Gamma, \qquad \int_{\Gamma_i} \mathbf{n}\cdot\boldsymbol{\psi} = 0 \ \ \textit{for } i = 0,\ldots,L. \tag{2.55}$$

Moreover, if Γ *is* $C^{1,1}$*, then* $\boldsymbol{\psi} \in H^1(\Omega)^3$*.*

Proof. See Theorems 3.4, 3.5, and 3.6 in Chapter I of [20]. ∎

THEOREM 2.5. *Let* $\Omega \subset \Re^3$ *satisfy Assumption A*1*. Assume that either* $\mathbf{n}\times\mathbf{w} = \mathbf{0}$ *on* Γ *or that* Ω *is simply connected. Then* $\mathbf{w} \in H(\mathbf{curl}\,;\Omega)$ *and* $\nabla\times\mathbf{w} = \mathbf{0}$ *if and only if* $\mathbf{w} = \nabla p$ *for some* $p \in H^1(\Omega)$ *and* p *is unique up to a constant.*

Proof. See Theorem 2.9 in Chapter I of [20]. ∎

Next, as in the two-dimensional proof, we provide a result that allows us to declare that a vector in $\mathbf{W}$ is $\mathbf{0}$.

LEMMA 2.7. *Let* A *be uniformly symmetric positive definite on a simply connected* $\Omega \subset \Re^3$*, which satisfies Assumptions A*1 *and A*2*. Let* $\mathbf{z} \in \mathbf{W}$ *satisfy*

$$\begin{aligned} &\text{i)} && \nabla^*\mathbf{z} &&= 0, && \textit{in } \Omega,\\ &\text{ii)} && \nabla\times A^{-1}\mathbf{z} &&= \mathbf{0}, && \textit{in } \Omega,\\ &\text{iii)} && \int_{\Gamma_i} \mathbf{n}\cdot\mathbf{z} &&= 0, && \textit{for } i \in D. \end{aligned} \tag{2.56}$$

Then $\mathbf{z} = \mathbf{0}$*.*

Proof. If $\Gamma_0 \subseteq \Gamma_N$, then $\mathbf{n}\cdot\mathbf{z} = 0$ on Γ_0 and we let $\widehat{N} = N \cup \{0\}$, $\widehat{D} = D$. If $\Gamma_0 \subseteq \Gamma_D$, let $\widehat{N} = N$, $\widehat{D} = D \cup \{0\}$. In either case, since $\mathbf{z} \in \mathbf{W}$, then assumptions (2.56) i) and iii) and Theorem 2.4 imply there exists $\boldsymbol{\psi} \in H^1(\Omega)^3$ such that $\mathbf{z} = \nabla\times\boldsymbol{\psi}$. Assumptions (2.56) ii) and Theorem 2.5 imply there exists $p \in H^1(\Omega)$ such that $A^{-1}\mathbf{z} = \nabla p$. Assumption (2.56) i) and Green's formula then yield

$$\begin{aligned} 0 &= (\nabla^*\mathbf{z},\ p)\\ &= (\mathbf{z},\ \nabla p) - \int_\Gamma (\mathbf{n}\cdot\mathbf{z})p\\ &= (A^{-1}\mathbf{z},\ \mathbf{z}) - \int_\Gamma (\mathbf{n}\cdot\mathbf{z})p. \end{aligned}$$

Since $\mathbf{z} \in \mathbf{W}$ implies $\mathbf{n}\cdot\mathbf{z} = 0$ on Γ_N, we then have

$$\begin{aligned} (A^{-1}\mathbf{z},\ \mathbf{z}) &= \sum_{i\in\widehat{D}} \int_{\Gamma_i} (\mathbf{n}\cdot\mathbf{z})p + \sum_{i\in\widehat{N}} \int_{\Gamma_i} (\mathbf{n}\cdot\mathbf{z})p\\ &= \sum_{i\in\widehat{D}} \int_{\Gamma_i} (\mathbf{n}\cdot\mathbf{z})p. \end{aligned}$$

Likewise, $\mathbf{z} \in \mathbf{W}$ implies $\mathbf{n} \times A^{-1}\mathbf{z} = \mathbf{0}$ on Γ_D, which by Lemma 2.6 yields

$$\int_{\Gamma_i} (\mathbf{n} \cdot \mathbf{z})p = \int_{\Gamma_i} (\mathbf{n} \cdot \nabla \times \boldsymbol{\psi})p = -\int_{\Gamma_i} (\mathbf{n} \times \boldsymbol{\psi}) \cdot \nabla p$$

$$= \int_{\Gamma_i} (\mathbf{n} \times \nabla p) \cdot \boldsymbol{\psi} = \int_{\Gamma_i} (\mathbf{n} \times A^{-1}\mathbf{z}) \cdot \boldsymbol{\psi} = 0$$

for $i \in \widehat{D}$. Thus,

$$(A^{-1}\mathbf{z},\ \mathbf{z}) = 0.$$

Since A is uniformly symmetric positive definite, it follows that $\mathbf{z} = \mathbf{0}$. ∎

The next result is a generalization of Theorem 2.4.

THEOREM 2.6. *Let $\Omega \subset \Re^3$ be simply connected and satisfy Assumptions A1 and A2. Then*

$$\mathbf{w} \in H(\mathrm{div};\, \Omega), \quad \nabla^* \mathbf{w} = 0, \quad \mathbf{n} \cdot \mathbf{w} = 0 \quad \text{on } \Gamma_D,$$

$$\int_{\Gamma_i} \mathbf{n} \cdot \mathbf{w} = 0 \quad \text{for } i \in N \tag{2.57}$$

if and only if there exists a unique $\boldsymbol{\psi} \in H(\mathbf{curl};\, \Omega)$ such that

$$\mathbf{w} = \nabla \times \boldsymbol{\psi}, \quad \nabla^* \boldsymbol{\psi} = 0, \quad \mathbf{n} \cdot \boldsymbol{\psi} = 0 \text{ on } \Gamma_N,$$

$$\mathbf{n} \times \boldsymbol{\psi} = \mathbf{0} \quad \text{on } \Gamma_D, \quad \int_{\Gamma_i} \mathbf{n} \cdot \boldsymbol{\psi} = 0 \quad \text{for } i \in D. \tag{2.58}$$

If Γ_0 is $C^{1,1}$, then $\boldsymbol{\psi} \in H^1(\Omega)^3$ and

$$\|\boldsymbol{\psi}\|_{1,\,\Omega} \le C \|\mathbf{w}\|_{0,\,\Omega}, \tag{2.59}$$

where C is a constant depending only on Ω.

Proof. First assume that (2.58) holds. Then, by Theorem 2.4a, $\mathbf{w} \in H(\mathrm{div};\, \Omega)$, $\nabla^* \mathbf{w} = 0$, and $\int_{\Gamma_i} \mathbf{n} \cdot \mathbf{w} = 0$ for $i = 1, \ldots, L$. Lemma 2.5 yields $\mathbf{n} \cdot \mathbf{w} = \mathbf{n} \cdot (\nabla \times \boldsymbol{\psi}) = 0$ on Γ_D.

Now assume that (2.57) holds. We will establish (2.58) and (2.59) together. First suppose that two functions $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ satisfy (2.58). Then it follows from Lemma 2.7 with $A = I$ that $\mathbf{z} = \boldsymbol{\psi}_1 - \boldsymbol{\psi}_2 = \mathbf{0}$, which proves the uniqueness. We must now construct $\boldsymbol{\psi} \in H(\mathbf{curl};\, \Omega)$ satisfying (2.58) and, when Γ_0 is $C^{1,1}$, (2.59). To apply Theorem 2.4c, we first extend $\mathbf{w}$ by $\mathbf{0}$ beyond the Dirichlet boundaries according to our two cases:

1) If $\Gamma_0 \subseteq \Gamma_N$, let $\widehat{N} = N \cup \{0\}$ and $\widehat{D} = D$, and define

$$\widehat{\Omega} = \Omega \cup \left(\cup_{i \in D} \overline{\Omega}_i \right).$$

Note that $\partial \widehat{\Omega} = \Gamma_N$.

2) If $\Gamma_0 \subseteq \Gamma_D$, let $\mathcal{O}$ be any bounded, open, simply connected set in $\Re^3$ with connected $C^{1,1}$ boundary such that $\overline{\Omega} \subset \mathcal{O}$. Let Ω_0 be the portion of $\mathcal{O}$ outside Γ_0 (see Fig. 2). Then let $\widehat{N} = N$ and $\widehat{D} = D \cup \{0\}$, and define

$$\widehat{\Omega} = \Omega_0 \cup \Gamma_0 \cup \Omega \cup \left(\cup_{i \in D} \overline{\Omega}_i \right).$$

Note that $\partial \widehat{\Omega} = \Gamma_N \cup \partial \mathcal{O}$.

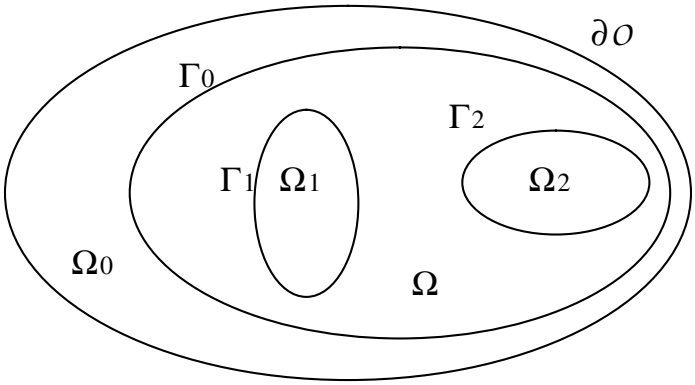

FIG. 2. *A section of* $\Omega \subset \Re^3$.

Now define

$$\widehat{\mathbf{w}} = \begin{cases} \mathbf{w}, & \text{in } \Omega, \\ \mathbf{0}, & \text{in } (\widehat{\Omega} \setminus \Omega) \cup (\partial\widehat{\Omega} \setminus \partial\Omega). \end{cases}$$

Since $\nabla^* \widehat{\mathbf{w}} \in L^2(\Omega_i)$ and $\mathbf{n} \cdot \widehat{\mathbf{w}}$ is continuous across each Γ_i for $i \in \widehat{D}$, then $\widehat{\mathbf{w}} \in H(\mathrm{div}; \widehat{\Omega})$ and $\nabla^* \widehat{\mathbf{w}} = 0$ in $\widehat{\Omega}$. In either case 1) or 2), $\int_{\Gamma_i} \mathbf{n} \cdot \widehat{\mathbf{w}} = 0$ on each piece of $\partial\widehat{\Omega}$ by assumption, so Theorem 2.4c implies that there exists a unique $\widehat{\boldsymbol{\psi}} \in H(\mathbf{curl}; \widehat{\Omega})$ such that

$$\widehat{\mathbf{w}} = \nabla \times \widehat{\boldsymbol{\psi}}, \quad \nabla^* \widehat{\boldsymbol{\psi}} = 0, \quad \mathbf{n} \cdot \widehat{\boldsymbol{\psi}} = 0 \ \text{ on } \partial\widehat{\Omega}, \tag{2.60}$$

and, if Γ_0 is $C^{1,1}$, then $\widehat{\boldsymbol{\psi}} \in H^1(\widehat{\Omega})^3$.

We now construct $\boldsymbol{\varphi} \in H(\mathbf{curl}; \widehat{\Omega})$ such that $\nabla^* \boldsymbol{\varphi} = 0$, $\nabla \times \boldsymbol{\varphi} = \mathbf{0}$ in $\widehat{\Omega}$, and $\mathbf{n} \times \boldsymbol{\varphi}$ matches $\mathbf{n} \times \widehat{\boldsymbol{\psi}}$ on Γ_D. Notice that $\nabla \times \widehat{\boldsymbol{\psi}} = \mathbf{0}$ in Ω_i for $i \in \widehat{D}$. Since Ω is simply connected, then each Ω_i is simply connected (cf. proof of Theorem 3.6 in Chapter I of [21]). By Theorem 2.5, $\widehat{\boldsymbol{\psi}} = \nabla q_i$ in Ω_i, where by (2.60) q_i satisfies

$$\begin{cases} \nabla^* \nabla q_i &= 0, & \text{in} \quad \Omega_i, \\ \mathbf{n} \cdot \nabla q_i &= \mathbf{n} \cdot \widehat{\boldsymbol{\psi}}, & \text{on} \quad \Gamma_i \end{cases} \tag{2.61}$$

for $i \in D$, which is a compatible Neumann problem in $H^1(\Omega_i)/\Re$.

If Γ_0 is $C^{1,1}$, then $\widehat{\boldsymbol{\psi}} \in H^1(\widehat{\Omega})^3$, which implies $\widehat{\boldsymbol{\psi}} \in H^1(\Omega_i)^3$. Together with the fact that Γ_i is $C^{1,1}$, we conclude that $\mathbf{n} \cdot \widehat{\boldsymbol{\psi}} \in H^{1/2}(\Gamma_i)$. Thus, (2.61) is $H^2(\Omega_i)$ regular and

$$\|q_i\|_{2,\Omega_i} \leq M_i \|\mathbf{n} \cdot \widehat{\boldsymbol{\psi}}\|_{1/2,\Gamma_i}, \tag{2.62}$$

where M_i is a constant depending only on Ω_i.

For case 2), we need to define the additional function q_0 that satisfies

$$\begin{cases} \nabla^* \nabla q_0 &= 0, & \text{in} \quad \Omega_0, \\ \mathbf{n} \cdot \nabla q_0 &= \mathbf{n} \cdot \widehat{\boldsymbol{\psi}}, & \text{on} \quad \Gamma_0 \cup \partial\mathcal{O}, \end{cases} \tag{2.63}$$

which is a compatible Neumann problem in $H^1(\Omega_0)/\Re$. If Γ_0 is $C^{1,1}$, then the same argument as above shows that (2.63) is $H^2(\Omega_0)$ regular and

$$\|q_0\|_{2,\Omega_0} \le M_0 \|\mathbf{n}\cdot\widehat{\boldsymbol{\psi}}\|_{1/2,\Gamma_0}, \tag{2.64}$$

where M_0 is a constant depending only on Ω_0.

Next, we choose q to satisfy

$$\begin{cases} \nabla^*\nabla q &= 0, & \text{in} & \Omega, \\ q &= q_i, & \text{on} & \Gamma_i,\ i\in\widehat{D}, \\ \mathbf{n}\cdot\nabla q &= 0, & \text{on} & \Gamma_N. \end{cases} \tag{2.65}$$

Then

$$\widehat{q} = \begin{cases} q, & \text{in } \Omega, \\ q_i, & \text{in } \Omega_i,\ i\in\widehat{D}, \end{cases}$$

is in $H^1(\widehat{\Omega})$, which implies that $\nabla\widehat{q}\in H(\mathbf{curl}\,;\widehat{\Omega})$, implying in turn that $\mathbf{n}\times\nabla q_i = \mathbf{n}\times\nabla q$ on Γ_D. Thus,

$$\mathbf{n}\times\widehat{\boldsymbol{\psi}} = \mathbf{n}\times\nabla q_i = \mathbf{n}\times\nabla q \tag{2.66}$$

on Γ_D.

If Γ_0 is $C^{1,1}$, then each $q_i\in H^2(\Omega_i)$, which implies that $q_i\in H^{3/2}(\Gamma_i)$, that problem (2.65) is $H^2(\Omega)$ regular, and, using (2.62) and (2.64), that there are constants $C_1 - C_3$ such that

$$\begin{aligned} \|q\|_{2,\Omega} &\le C_1\sum_{i\in\widehat{D}}\|q_i\|_{3/2,\Gamma_i} \le C_2\sum_{i\in\widehat{D}}\|q_i\|_{2,\Omega_i} \\ &\le C_3\sum_{i\in\widehat{D}}\|\mathbf{n}\cdot\widehat{\boldsymbol{\psi}}\|_{1/2,\Gamma_i} \le C_3\|\widehat{\boldsymbol{\psi}}\|_{1,\Omega}. \end{aligned} \tag{2.67}$$

Consider the functions p_i for $i\in\widehat{D}$ that satisfy

$$\begin{cases} \nabla^*\nabla p_i &= 0, & \text{in} & \Omega, \\ p_i &= 1, & \text{on} & \Gamma_i, \\ p_i &= 0, & \text{on} & \Gamma_D\setminus\Gamma_i, \\ \mathbf{n}\cdot\nabla p_i &= 0, & \text{on} & \Gamma_N. \end{cases} \tag{2.68}$$

Clearly, (2.68) is $H^2(\Omega)$ regular and

$$\|p_i\|_{2,\Omega} \le N_i \tag{2.69}$$

for some constants N_i depending only on Ω.

We now set

$$\boldsymbol{\varphi} = \nabla q + \sum_{i\in\widehat{D}}\alpha_i\nabla p_i, \tag{2.70}$$

where the constants α_i are chosen so that

$$\int_{\Gamma_i}\mathbf{n}\cdot(\widehat{\boldsymbol{\psi}}-\boldsymbol{\varphi}) = 0 \quad \text{for } i\in\widehat{D}. \tag{2.71}$$

It is easy to see that $\boldsymbol{\varphi}$ is divergence and curl free in Ω, i.e.,

$$\nabla^* \boldsymbol{\varphi} = 0 \quad \text{and} \quad \nabla \times \boldsymbol{\varphi} = \mathbf{0}, \quad \text{in } \Omega. \tag{2.72}$$

Now the α_i are determined by solving the matrix equation

$$\sum_{j \in \widehat{D}} \left(\int_{\Gamma_i} \mathbf{n} \cdot \nabla p_j \right) \alpha_j = \int_{\Gamma_i} \mathbf{n} \cdot (\widehat{\boldsymbol{\psi}} - \nabla q), \tag{2.73}$$

for $i \in \widehat{D}$. Thus, $\widehat{\boldsymbol{\psi}} - \nabla q$ and each ∇p_j are divergence free and their normal components vanish on Γ_N, so the integrals of their normal components over the entire boundary are zero. This implies that each column of the system sums to zero, as does the right-hand side. Hence, it is a consistent but singular system, with a null space that contains constant solutions, which yield a zero of ∇. Now if we delete any column and corresponding row, setting the corresponding α_i to zero, then Lemma 2.7 with $A = I$ implies that this reduced system is nonsingular: the difference of any two functions arising from (2.70) and (2.73) satisfies the hypotheses of Lemma 2.7, so that the difference is zero; this implies that any solution of the reduced system must be unique; since the system is square, it must be nonsingular.

Since the left-hand side of (2.73) depends only on Ω (see (2.69)), then there is a constant C_4 depending only on Ω such that

$$\max_{i \in \widehat{D}} |\alpha_i| \leq C_4 \sum_{i \in \widehat{D}} \left| \int_{\Gamma_i} \mathbf{n} \cdot (\widehat{\boldsymbol{\psi}} - \nabla q) \right| \leq C_4 \sum_{i \in \widehat{D}} \| \mathbf{n} \cdot (\widehat{\boldsymbol{\psi}} - \nabla q) \|_{-1/2, \Gamma_i}$$

$$\leq C_4 \| \widehat{\boldsymbol{\psi}} - \nabla q \|_{H(\mathrm{div}; \Omega)} = C_4 \| \widehat{\boldsymbol{\psi}} - \nabla q \|_{0, \Omega}. \tag{2.74}$$

We now define

$$\boldsymbol{\psi} = \widehat{\boldsymbol{\psi}} - \boldsymbol{\varphi}. \tag{2.75}$$

To see that $\boldsymbol{\psi}$ satisfies (2.58), note first that (2.60) and (2.72) imply

$$\nabla^* \boldsymbol{\psi} = 0 \quad \text{and} \quad \nabla \times \boldsymbol{\psi} = \mathbf{w} \quad \text{in } \Omega.$$

Equation (2.71) implies that

$$\int_{\Gamma_i} \mathbf{n} \cdot \boldsymbol{\psi} = 0$$

for $i \in \widehat{D}$. Finally, note that (2.60), (2.65), (2.68), and (2.70) yield

$$\mathbf{n} \cdot \boldsymbol{\psi} = \mathbf{n} \cdot \widehat{\boldsymbol{\psi}} - \mathbf{n} \cdot \boldsymbol{\varphi} = 0$$

on Γ_N, while (2.70), (2.68), and (2.66) yield

$$\mathbf{n} \times \boldsymbol{\psi} = \mathbf{n} \times \widehat{\boldsymbol{\psi}} - \mathbf{n} \times \nabla q = \mathbf{0}$$

on Γ_D.

If Γ_0 is $C^{1,1}$, then, as was mentioned above, $\widehat{\boldsymbol{\psi}} \in H^1(\widehat{\Omega})^3$. Now, $\widehat{\boldsymbol{\psi}}$ and $\widehat{\Omega}$ were constructed so that $\widehat{\boldsymbol{\psi}} \in H(\mathrm{div}, \widehat{\Omega}) \cap H(\mathbf{curl}, \widehat{\Omega})$ with $\mathbf{n} \cdot \widehat{\boldsymbol{\psi}} = 0$ on $\partial\widehat{\Omega}$. Thus, Theorem 3.9 in Chapter I of [20] applies and yields

$$\begin{aligned} \|\widehat{\boldsymbol{\psi}}\|_{1,\widehat{\Omega}} &\le C_5(\|\widehat{\boldsymbol{\psi}}\|_{0,\widehat{\Omega}} + \|\nabla^*\widehat{\boldsymbol{\psi}}\|_{0,\widehat{\Omega}} + \|\nabla\times\widehat{\boldsymbol{\psi}}\|_{0,\widehat{\Omega}}) \\ &= C_5(\|\widehat{\boldsymbol{\psi}}\|_{0,\widehat{\Omega}} + \|\nabla\times\widehat{\boldsymbol{\psi}}\|_{0,\widehat{\Omega}}), \end{aligned} \tag{2.76}$$

for some constant C_5 depending only on $\widehat{\Omega}$. Lemma 3.6 in Chapter I of [20] yields

$$\|\widehat{\boldsymbol{\psi}}\|_{0,\widehat{\Omega}} \le C_6\|\nabla\times\widehat{\boldsymbol{\psi}}\|_{0,\widehat{\Omega}} \tag{2.77}$$

for some constant C_6 depending only on $\widehat{\Omega}$. Since $\nabla\times\widehat{\boldsymbol{\psi}} = \mathbf{0}$ in $\widehat{\Omega}\setminus\Omega$, then (2.76) and (2.77) yield

$$\|\widehat{\boldsymbol{\psi}}\|_{1,\Omega} \le \|\widehat{\boldsymbol{\psi}}\|_{1,\widehat{\Omega}} \le C_7\|\nabla\times\widehat{\boldsymbol{\psi}}\|_{0,\widehat{\Omega}} = C_7\|\mathbf{w}\|_{0,\Omega}. \tag{2.78}$$

We remark that because of the manner in which $\widehat{\Omega}$ was chosen, the constants C_5–C_7 can be considered to depend only on Ω.

We also have from (2.70), (2.69), (2.74), and (2.67) that

$$\begin{aligned} \|\boldsymbol{\varphi}\|_{1,\Omega} &\le \|\nabla q\|_{1,\Omega} + \sum_{i\in\widehat{D}} |\alpha_i| \|\nabla p_i\|_{1,\Omega} \\ &\le \|\nabla q\|_{1,\Omega} + C_8\|\widehat{\boldsymbol{\psi}} - \nabla q\|_{0,\Omega} \\ &\le C_9\|q\|_{2,\Omega} + C_8\|\widehat{\boldsymbol{\psi}}\|_{0,\Omega} \\ &\le C_{10}\|\widehat{\boldsymbol{\psi}}\|_{1,\Omega}, \end{aligned} \tag{2.79}$$

where C_8–C_{10} depend only on Ω. Equations (2.78) and (2.79) yield

$$\|\boldsymbol{\psi}\|_{1,\Omega} \le \|\widehat{\boldsymbol{\psi}}\|_{1,\Omega} + \|\boldsymbol{\varphi}\|_{1,\Omega} \le C\,\|\mathbf{w}\|_{0,\Omega},$$

which proves the theorem. □

Proof of Theorem 2.2. To prove the lower bound in (2.23), we first restrict Ω to be simply connected and Γ_0 to be $C^{1,1}$, and define $\widehat{D}$ and $\widehat{N}$ as in the proof of Theorem 2.6.

Let $\mathbf{v} \in \mathbf{W}$ and define $\mathbf{w} = \nabla\times A^{-1}\mathbf{v}$. By Theorem 2.4, $\mathbf{w} \in H(\mathrm{div};\,\Omega)$ and

$$\nabla^*\mathbf{w} = 0, \quad \int_{\Gamma_i} \mathbf{n}\cdot\mathbf{w} = 0 \quad \text{for } i = 0, \ldots, L.$$

Since $\mathbf{v} \in \mathbf{W}$, then $A^{-1}\mathbf{v} \in H(\mathbf{curl};\,\Omega)$ and $\mathbf{n} \times A^{-1}\mathbf{v} = \mathbf{0}$ on Γ_D, so Lemma 2.5 applied to $A^{-1}\mathbf{v}$ yields

$$\mathbf{n}\cdot\mathbf{w} = \mathbf{n}\cdot\nabla\times A^{-1}\mathbf{v} = 0$$

on Γ_D. Thus, $\mathbf{w}$ satisfies hypothesis (2.57) of Theorem 2.6, so there exists $\boldsymbol{\psi} \in H^1(\Omega)^3$ such that

$$\begin{gathered} \mathbf{w} = \nabla\times\boldsymbol{\psi}, \quad \nabla^*\boldsymbol{\psi} = 0, \quad \mathbf{n}\cdot\boldsymbol{\psi} = 0, \text{ on } \Gamma_N, \\ \mathbf{n}\times\boldsymbol{\psi} = \mathbf{0} \text{ on } \Gamma_D, \quad \int_{\Gamma_i} \mathbf{n}\cdot\boldsymbol{\psi} = 0, \quad \text{for } i \in D. \end{gathered} \tag{2.80}$$

Moreover,

$$\|\boldsymbol{\psi}\|_{1,\Omega} \le C_1 \|\mathbf{w}\|_{0,\Omega} = C_1 \|\nabla \times A^{-1}\mathbf{v}\|_{0,\Omega}. \tag{2.81}$$

Next, consider $A^{-1}\mathbf{v} - \boldsymbol{\psi}$, which is in $H(\mathbf{curl}\,;\Omega)$. Since $\nabla\times(A^{-1}\mathbf{v} - \boldsymbol{\psi}) = \mathbf{0}$, then Theorem 2.5 implies that $A^{-1}\mathbf{v} - \boldsymbol{\psi} = \nabla p$ for some $p \in H^1(\Omega)$. We first construct p, then bound it. Let p_0 satisfy

$$\begin{cases} \nabla^* A \nabla p_0 &= \nabla^*\mathbf{v} - \nabla^* A\boldsymbol{\psi}, & \text{in} \quad \Omega, \\ p_0 &= 0, & \text{on} \quad \Gamma_D, \\ \mathbf{n}\cdot A\nabla p_0 &= -\mathbf{n}\cdot A\boldsymbol{\psi}, & \text{on} \quad \Gamma_N. \end{cases} \tag{2.82}$$

By hypothesis, problem (2.82) is $H^2(\Omega)$ regular and, using (2.81), we have

$$\begin{aligned} \|p_0\|_{2,\Omega} &\le C_2(\|\nabla^*\mathbf{v}\|_{0,\Omega} + \|\nabla^* A\boldsymbol{\psi}\|_{0,\Omega} + \|\mathbf{n}\cdot A\boldsymbol{\psi}\|_{1/2,\Gamma_N}) \\ &\le C_2\,\|\nabla^*\mathbf{v}\|_{0,\Omega} + C_3\|\boldsymbol{\psi}\|_{1,\Omega} \\ &\le C_2\,\|\nabla^*\mathbf{v}\|_{0,\Omega} + C_4\|\nabla\times A^{-1}\mathbf{v}\|_{0,\Omega}. \end{aligned} \tag{2.83}$$

As in the proof of Theorem 2.6, consider the functions p_i for $i \in D$ that satisfy

$$\begin{cases} \nabla^* A\nabla p_i &= 0, & \text{in} \quad \Omega, \\ p_i &= 1, & \text{on} \quad \Gamma_i, \\ p_i &= 0, & \text{on} \quad \Gamma_D \setminus \Gamma_i, \\ \mathbf{n}\cdot A\nabla p_i &= 0, & \text{on} \quad \Gamma_N. \end{cases} \tag{2.84}$$

Clearly, (2.84) is $H^2(\Omega)$ regular and

$$\|p_i\|_{2,\Omega} \le N_i \tag{2.85}$$

for some constants N_i depending only on Ω and A.

Now set

$$p = p_0 + \sum_{i\in D} \alpha_i p_i, \tag{2.86}$$

and choose the α_i so that

$$\int_{\Gamma_i} \mathbf{n}\cdot(\mathbf{v} - A\nabla p - A\boldsymbol{\psi}) = 0 \quad \text{for } i \in D. \tag{2.87}$$

This yields the matrix equation

$$\sum_{j\in D}\left(\int_{\Gamma_i} A\nabla p_j\right)\alpha_j = \int_{\Gamma_i} \mathbf{n}\cdot(\mathbf{v} - A\nabla p_0 - A\boldsymbol{\psi}) \quad \text{for } i \in D. \tag{2.88}$$

Similar to the proof of Theorem 2.6, this system has a solution that is either unique (when $D \ne \widehat{D}$) or unique up to a constant (when $D = \widehat{D}$). The matrix of (2.88) depends only on Ω and A, so there is a constant C_5 depending only on Ω and A such

that

$$\begin{aligned}\max_{i\in\widehat{D}}|\alpha_i| &\le C_5\sum_{i\in\widehat{D}}\left|\int_{\Gamma_i}\mathbf{n}\cdot(\mathbf{v}-A\nabla p_0-A\boldsymbol{\psi})\right| \\ &\le C_5\sum_{i\in\widehat{D}}\|\mathbf{n}\cdot(\mathbf{v}-A\nabla p_0-A\boldsymbol{\psi})\|_{-1/2,\,\Gamma_i} \\ &\le C_5\|\mathbf{v}-A\nabla p_0-A\boldsymbol{\psi}\|_{H(\mathrm{div};\,\Omega)} \\ &\le C_5\|\mathbf{v}\|_{H(\mathrm{div};\,\Omega)}+C_6\|p_0\|_{2,\,\Omega}+C_7\|\boldsymbol{\psi}\|_{1,\,\Omega}. \qquad (2.89)\end{aligned}$$

To see that $\mathbf{v}=A\nabla p+A\boldsymbol{\psi}$, set

$$\mathbf{z}=\mathbf{v}-A\nabla p-A\boldsymbol{\psi},$$

and note from (2.80) that

$$\nabla\times A^{-1}\mathbf{z}=\nabla\times A^{-1}\mathbf{v}-\nabla\times\boldsymbol{\psi}=\mathbf{0}$$

and from (2.82) that

$$\nabla^*\mathbf{z}=\nabla^*\mathbf{v}-\nabla^*A\nabla p_0-\nabla^*A\boldsymbol{\psi}=0.$$

Since $\mathbf{v}\in\mathbf{W}$ implies $\mathbf{n}\cdot\mathbf{v}=0$ on Γ_N, then (2.82) and (2.84) yield

$$\begin{aligned}\mathbf{n}\cdot\mathbf{z}&=\mathbf{n}\cdot\mathbf{v}-\mathbf{n}\cdot A\nabla p-\mathbf{n}\cdot A\boldsymbol{\psi}\\ &=-\mathbf{n}\cdot A\nabla p_0-\mathbf{n}\cdot A\boldsymbol{\psi}=0\end{aligned}$$

on Γ_N. Likewise, $\mathbf{v}\in\mathbf{W}$ implies $\mathbf{n}\times A^{-1}\mathbf{v}=\mathbf{0}$ on Γ_D and (2.82) and (2.84) imply $\mathbf{n}\times\nabla p=\mathbf{0}$ on Γ_D. Together with (2.80) we then have

$$\mathbf{n}\times A^{-1}\mathbf{z}=\mathbf{n}\times A^{-1}\mathbf{v}-\mathbf{n}\times\nabla p-\mathbf{n}\times\boldsymbol{\psi}=\mathbf{0}$$

on Γ_D. Finally, with (2.87) we see that $\mathbf{z}$ satisfies the hypotheses of Lemma 2.7, so $\mathbf{z}=\mathbf{0}$ and, hence, $\mathbf{v}=A\nabla p+A\boldsymbol{\psi}$.

The bounds (2.85), (2.89), (2.81), and (2.83) yield

$$\begin{aligned}\|\mathbf{v}\|_{1,\,\Omega}&\le\|A\nabla p_0\|_{1,\,\Omega}+\sum_{i\in D}|\alpha_i|\|A\nabla p_i\|_{1,\,\Omega}+\|A\boldsymbol{\psi}\|_{1,\,\Omega}\\ &\le\|A\nabla p_0\|_{1,\,\Omega}+C_8(\|\mathbf{v}\|_{H(\mathrm{div};\,\Omega)}+\|p_0\|_{2,\,\Omega}+\|\boldsymbol{\psi}\|_{1,\,\Omega})+\|A\boldsymbol{\psi}\|_{1,\,\Omega}\\ &\le C_8\|\mathbf{v}\|_{H(\mathrm{div};\,\Omega)}+C_9\|p_0\|_{2,\,\Omega}+C_{10}\|\boldsymbol{\psi}\|_{1,\,\Omega}\\ &\le C(\|\mathbf{v}\|_{0,\,\Omega}+\|\nabla^*\mathbf{v}\|_{0,\,\Omega}+\|\nabla\times A^{-1}\mathbf{v}\|_{0,\,\Omega})\end{aligned}$$

for some constants C_8-C_{10} and C depending only on Ω and A. This proves the theorem for simply connected Ω and smooth Γ_0.

The proof for multiply-connected regions follows from a partition of unity argument as in part 2) of the proof of Theorem 3.7 in Chapter I of [21]. The proof when Γ_0 is a convex polyhedron is analogous to the proof for this case offered in Theorem 3.9 in Chapter I of [20]. There, a sequence of subregions $\Omega_j\subseteq\Omega$ with $C^{1,1}$ boundaries is constructed to converge outward to Ω. Using the fact that the result holds on each Ω_j, the result is shown to hold on Ω. □

In the remainder of this paper, we will assume that the conclusion of Theorem 2.2 holds.

3. Finite element approximation. Using two-dimensional terminology in these last two sections, to discretize the least-squares variational form (2.14), let T_h be a regular triangulation of Ω with elements of size $O(h)$ satisfying the inverse assumption (see [18]). Assume we are given two finite element approximation subspaces

$$\mathbf{W}_h \subset \mathbf{W} \quad \text{and} \quad V_h \subset V$$

defined on the triangulation T_h. Then the finite element approximation to (2.14) is to find $(\mathbf{u}_h, p_h) \in \mathbf{W}_h \times V_h$ such that

$$\mathcal{F}(\mathbf{u}_h, p_h; \mathbf{v}, q) = f(\mathbf{v}, q) \quad \forall\, (\mathbf{v}, q) \in \mathbf{W}_h \times V_h. \tag{3.1}$$

For simplicity, we only consider continuous piecewise linear finite element spaces; i.e.,

$$V_h = \{q \in C^0(\Omega) \,:\, q|_K \in P_1(K) \quad \forall\, K \in T_h,\ q \in V\}$$

and

$$\mathbf{W}_h = \{\mathbf{v} \in C^0(\Omega)^n \,:\, v_l|_K \in P_1(K) \quad \forall\, K \in T_h,\ \mathbf{v} \in \mathbf{W}\},$$

where $P_1(K)$ is the space of polynomials of degree at most one. Extension of the following results to higher-order finite element approximation spaces is straightforward. (See [11] for the more general case and for the proofs of both theorems of this section.)

THEOREM 3.1. *Assume that the solution,* $(\mathbf{u}, p)$*, of* (2.14) *is in* $H^{1+\alpha}(\Omega)^{n+1}$ *for some* $\alpha \in [0, 1]$*, and let* $(\mathbf{u}_h, p_h) \in \mathbf{W}_h \times V_h$ *be the solution of* (3.1)*. Then*

$$\|p - p_h\|_{1,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \le C\, h^{\alpha} \left(\|p\|_{1+\alpha,\Omega} + \|\mathbf{u}\|_{1+\alpha,\Omega}\right), \tag{3.2}$$

where the constant C *does not depend on* h*,* p*, or* $\mathbf{u}$*.*

Proof. The proof is a direct result of Theorem 2.2, Céa's lemma, and interpolation properties of piecewise linear functions (cf. [18]). □

Let $\{\phi_1, \ldots, \phi_N\}$ and $\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_M\}$ be bases for V_h and $\mathbf{W}_h$, respectively. Then, for any $q \in V_h$ and $\mathbf{v} \in \mathbf{W}_h$, we have

$$q = \sum_{i=1}^{N} \eta_i \phi_i \quad \text{and} \quad \mathbf{v} = \sum_{i=1}^{M} \xi_i \boldsymbol{\psi}_i.$$

Let $|\boldsymbol{\eta}|$ and $|\boldsymbol{\xi}|$ denote the respective l_2-norms of the vectors $\boldsymbol{\eta} = (\ldots, \eta_i, \ldots)^T$ and $\boldsymbol{\xi} = (\ldots, \xi_i, \ldots)^T$. Assume that there exist positive constants β_i $(i = 0, 1, 2, 3)$ such that

$$\beta_0\, h^n\, |\boldsymbol{\eta}| \le \|q\|_{0,\Omega} \le \beta_1\, h^n\, |\boldsymbol{\eta}| \tag{3.3}$$

and

$$\beta_2\, h^n\, |\boldsymbol{\xi}| \le \|\mathbf{v}\|_{0,\Omega} \le \beta_3\, h^n\, |\boldsymbol{\xi}|. \tag{3.4}$$

Standard finite element spaces with the usual bases satisfy these bounds.

THEOREM 3.2. *The condition number of the linear system resulting from* (3.1) *is* $O(h^{-2})$*.*

Proof. The proof is the same as the proof of Theorem 6.1 in [11]. □

Remark 3.1. *The proof of Theorem* 6.1 *in* [11] *shows that the smallest and largest eigenvalues of the matrices arising from* (3.1) *are* $O(h^n)$ *and* $O(h^{n-2})$*, respectively. It is natural to scale the matrix by multiplying it by a factor of order* h^{-n}*, so that the smallest and largest eigenvalues are* $O(1)$ *and* $O(h^{-2})$*, respectively. In subsequent sections, we assume that the matrix arising from* (3.1) *and the operator associated with the bilinear form* $\mathcal{F}(\cdot\,;\,\cdot)$ *are so scaled.*

4. Multilevel additive and multiplicative algorithms. To develop multilevel solution methods for matrix equations arising from problem (3.1), we start with a coarse grid triangulation T_0 of $\bar{\Omega}$ having the property that the boundary Γ is composed of edges of triangles $K \in T_0$ and points of $\bar{\Gamma}_D \cap \bar{\Gamma}_N$ are included as nodes of T_0. T_0 is then regularly refined several times, giving a family of nested triangulations

$$T_0, T_1, \ldots, T_J = T_h$$

such that every triangle of T_{j+1} is either a triangle of T_j or is generated by subdividing a triangle of T_j into some congruent triangles (cf. [18]). Let h_j denote the mesh size of triangulation T_j and assume that

$$1 < \gamma \leq \gamma_j \equiv \frac{h_j}{h_{j+1}} \leq C$$

for fixed constants γ and C and for $j = 0, 1, \ldots, J-1$. (Here and henceforth, we will use C with or without subscripts to denote a generic positive constant independent of the mesh parameter h and the number of levels J. We will also use subscripts like j in place of the more cumbersome h_j so that V_j is used in place of V_{h_j}, for example.) Without loss of generality, we assume more specifically that

$$\gamma = \gamma_j = 2, \quad j = 0, 1, \ldots, J-1.$$

For each $j = 0, 1, \ldots, J$, we associate the triangulation T_j with the continuous piecewise linear finite element space $\mathbf{W}_j \times V_j$. It is easy to verify that the family of spaces $\{\mathbf{W}_j \times V_j\}$ is nested, i.e.,

$$\mathbf{W}_0 \times V_0 \subset \mathbf{W}_1 \times V_1 \subset \cdots \subset \mathbf{W}_J \times V_J = \mathbf{W}_h \times V_h.$$

Our multigrid algorithms are described in terms of certain auxiliary operators. Fixing $j \in \{0, 1, \ldots, J\}$, define the (self-adjoint) level j operator $\mathcal{F}_j : \mathbf{W}_j \times V_j \longrightarrow \mathbf{W}_j \times V_j$ for $(\mathbf{u}, p) \in \mathbf{W}_j \times V_j$ by

$$(\mathcal{F}_j(\mathbf{u}, p); \mathbf{v}, q) = \mathcal{F}(\mathbf{u}, p; \mathbf{v}, q) \quad \forall\, (\mathbf{v}, q) \in \mathbf{W}_j \times V_j,$$

where the inner product $(\cdot\,; \cdot)$ is defined by

$$(\mathbf{u}, p; \mathbf{v}, q) = (\mathbf{u}, \mathbf{v})_{0,\Omega} + (p, q)_{0,\Omega}.$$

Also, define the respective elliptic and L^2 projection operators $\mathcal{P}_j, \mathcal{Q}_j : \mathbf{W}_h \times V_h \longrightarrow \mathbf{W}_j \times V_j$ for $(\mathbf{u}, p) \in \mathbf{W}_h \times V_h$ by

$$\mathcal{F}(\mathcal{P}_j(\mathbf{u}, p); \mathbf{v}, q) = \mathcal{F}(\mathbf{u}, p; \mathbf{v}, q) \quad \forall\, (\mathbf{v}, q) \in \mathbf{W}_j \times V_j$$

and

$$(\mathcal{Q}_j(\mathbf{u}, p); \mathbf{v}, q) = (\mathbf{u}, p; \mathbf{v}, q) \quad \forall\, (\mathbf{v}, q) \in \mathbf{W}_j \times V_j.$$

It is easy to verify that

$$\mathcal{Q}_j \mathcal{F}_J = \mathcal{F}_j \mathcal{P}_j. \tag{4.1}$$

Let $\mathcal{R}_0 = \mathcal{F}_0^{-1}$ and let $\mathcal{R}_j : \mathbf{W}_j \times V_j \longrightarrow \mathbf{W}_j \times V_j$ for $1 \leq j \leq J$ denote the smoothing operators. Below we specify certain general conditions on $\mathcal{R}_j$. For this purpose, we define λ_j to be the largest eigenvalue (or spectral radius) of $\mathcal{F}_j$ for $1 \leq j \leq J$.

Problem (3.1) can be rewritten as

$$\mathcal{F}_J(\mathbf{u}_J, p_J) = F_J, \tag{4.2}$$

where F_J is the right-hand side vector. The *additive* multigrid preconditioner is defined by

$$\mathcal{G}_J = \sum_{j=0}^{J} \mathcal{R}_j \mathcal{Q}_j. \tag{4.3}$$

The effectiveness of $\mathcal{G}_J$ is estimated by the condition number of the preconditioned operator

$$\mathcal{G}_J \mathcal{F}_J = \sum_{j=0}^{J} \mathcal{T}_j, \tag{4.4}$$

where

$$\mathcal{T}_j = \mathcal{R}_j \mathcal{Q}_j \mathcal{F}_J.$$

For the *multiplicative* multigrid scheme, we define the following.

ALGORITHM 4.1 (Multiplicative multigrid algorithm). *Given the initial approximation* $(\mathbf{u}_J^{old}, p_J^{old})$, *perform the following steps:*

1. *Letting* $z_J = (\mathbf{u}_J^{old}, p_J^{old})$, *compute an update from each successively coarser level according to*

$$z_{J-j-1} = z_{J-j} + \mathcal{R}_{J-j}\mathcal{Q}_{J-j}(F_J - \mathcal{F}_J z_{J-j}) \quad \textit{for} \quad j = 0, 1, \ldots, J-1.$$

2. *Letting* $\hat{z}_0 = z_0$, *compute an update from each successively finer level according to*

$$\hat{z}_j = \hat{z}_{j-1} + \mathcal{R}_{j-1}\mathcal{Q}_{j-1}(F_J - \mathcal{F}_J \hat{z}_{j-1}) \quad \textit{for} \quad j = 1, 2, \ldots, J.$$

3. *The final approximation is*

$$(\mathbf{u}_J^{new}, p_J^{new}) = \hat{z}_J.$$

Letting $(\mathbf{u}_J, p_J)$ denote the exact solution of (4.2), then it is easy to verify that the multiplicative multigrid error equation for the new approximation $(\mathbf{u}_J^{new}, p_J^{new})$ is given by

$$(\mathbf{u}_J, p_J) - (\mathbf{u}_J^{new}, p_J^{new}) = \mathcal{E}_J^s \left((\mathbf{u}_J, p_J) - (\mathbf{u}_J^{old}, p_J^{old}) \right),$$

where the least-squares error reduction operator is given by

$$\mathcal{E}_J^s = \mathcal{E}_J \mathcal{E}_J^*,$$

with

$$\begin{aligned} \mathcal{E}_J &= (\mathcal{I} - \mathcal{T}_J)(\mathcal{I} - \mathcal{T}_{J-1}) \cdots (\mathcal{I} - \mathcal{T}_0), \\ \mathcal{E}_J^* &= (\mathcal{I} - \mathcal{T}_0^*) \cdots (\mathcal{I} - \mathcal{T}_{J-1}^*)(\mathcal{I} - \mathcal{T}_J^*). \end{aligned} \tag{4.5}$$

(Here we use the fact that the coarsest grid uses an exact solver, so that $\mathcal{I} - \mathcal{T}_0 = (\mathcal{I} - \mathcal{T}_0)^2$.) Note that $\mathcal{E}_J^*$ is the adjoint of $\mathcal{E}_J$ with respect to the inner product $\mathcal{F}(\cdot\,;\,\cdot)$. Since the error reduction operator $\mathcal{E}_J^s$ is self-adjoint with respect to the inner product $\mathcal{F}(\cdot\,;\,\cdot)$, we have

$$\|\mathcal{E}_J^s\|_{\mathcal{F}} = \|\mathcal{E}_J\|_{\mathcal{F}}^2, \tag{4.6}$$

where the *$\mathcal{F}$-norm* is defined by

$$\|\cdot\|_{\mathcal{F}} = \sqrt{\mathcal{F}(\cdot\,;\,\cdot)}.$$

Remark 4.1. *Algorithm* 4.1 *is a conventional $V(1,1)$-cycle multigrid algorithm applied to problem* (4.2)*. More general multigrid algorithms, including nonsymmetric cycling schemes like $V(1,0)$-cycles and other cycling schemes like W-cycles, may be defined* (*see* [22] *and* [26])*, but we restrict ourselves to the above algorithm for concreteness and because the results are most interesting for this case. The following immediately extends to such schemes, although the W-cycle results are not stronger* (*see* [6])*. Moreover, our optimal V-cycle theory can be used in the usual way* (*cf.* [26]) *to establish optimal* total *complexity of full-multigrid V-cycles (FMV). This is straightforward but nevertheless important: while the V-cycle is viewed as an iterative method that obtains optimal algebraic convergence* factors*, an FMV algorithm based on such cycles is essentially a direct method that achieves overall accuracy to the level of discretization error at a total cost proportional to the number of unknowns.*

Letting

$$\begin{aligned}\mathcal{G}(\mathbf{u},\,p;\,\mathbf{v},\,q) = (\mathbf{u},\,p;\,\mathbf{v},\,q) + (\nabla^*\mathbf{u},\,\nabla^*\mathbf{v})_{0,\,\Omega}\\ +(\mathbf{curl}\,(A^{-1}\mathbf{u}),\,\mathbf{curl}\,(A^{-1}\mathbf{v}))_{0,\,\Omega} + (\nabla p,\,\nabla q)_{0,\,\Omega}\end{aligned}$$

and

$$(\mathbf{u},\,p;\,\mathbf{v},\,q)_{1,\,\Omega} = (\mathbf{u},\,p;\,\mathbf{v},\,q) + (\nabla\mathbf{u},\,\nabla\mathbf{v})_{0,\,\Omega} + (\nabla p,\,\nabla q)_{0,\,\Omega}, \tag{4.7}$$

by Theorems 2.1 and 2.2 we know that the bilinear forms $\mathcal{F}(\cdot\,;\,\cdot)$ and $\mathcal{G}(\cdot\,;\,\cdot)$ and the inner product $(\cdot\,;\,\cdot)_{1,\,\Omega}$ are uniformly equivalent on $\mathbf{W}_h \times V_h$ with respect to h and J; i.e., there exist positive constants C_i $(i = 0,\,1,\,2,\,3)$ such that, for any $(\mathbf{v},\,q) \in \mathbf{W}_h \times V_h$,

$$\begin{aligned}C_0\,(\mathbf{v},\,q;\,\mathbf{v},\,q)_{1,\,\Omega} \le C_1\,\mathcal{G}(\mathbf{v},\,q;\,\mathbf{v},\,q) \le \mathcal{F}(\mathbf{v},\,q;\,\mathbf{v},\,q)\\ \le C_2\,\mathcal{G}(\mathbf{v},\,q;\,\mathbf{v},\,q) \le C_3\,(\mathbf{v},\,q;\,\mathbf{v},\,q)_{1,\,\Omega}.\end{aligned} \tag{4.8}$$

Optimality of the additive scheme is a direct consequence of this equivalence and existing multigrid theory (cf. [28, 29, 5]). We state it without proof.

THEOREM 4.1 (Additive multigrid algorithm). *For any $(\mathbf{v},\,q) \in \mathbf{W}_h \times V_h$, assume that $\mathcal{R}_j$ is a symmetric operator with respect to $(\cdot\,;\,\cdot)$ and that it satisfies*

$$C_1\,\frac{\|(\mathbf{v},\,q)\|_{0,\,\Omega}^2}{\lambda_j} \le (\mathcal{R}_j(\mathbf{v},\,q);\,\mathbf{v},\,q) \le C_2\,\frac{\|(\mathbf{v},\,q)\|_{0,\,\Omega}^2}{\lambda_j} \quad \textit{for } j = 1,\,2,\,\ldots,\,J.$$

Then

$$C_1\,\mathcal{F}(\mathbf{v},\,q;\,\mathbf{v},\,q) \le \mathcal{F}(\mathcal{G}_J\mathcal{F}_J(\mathbf{v},\,q);\,\mathbf{v},\,q) \le C_2\,\mathcal{F}(\mathbf{v},\,q;\,\mathbf{v},\,q). \tag{4.9}$$

Although multiplicative multigrid is often more efficient than the additive scheme, its convergence theory can be more complicated. We will establish convergence of multiplicative multigrid by verifying conditions required by the theory developed in [5] (see also [36]), the only nontrivial task of which is to verify their second assumption on the smoother as the next lemma does.

LEMMA 4.1. *Let* $\tilde{\mathcal{R}}_0 = \mathcal{F}_0^{-1}$, $\tilde{\mathcal{R}}_j = \frac{1}{\lambda_j}\mathcal{I}$ *for* $j > 0$, *where* $\mathcal{I}$ *is the identity operator, and* $\tilde{\mathcal{T}}_j = \tilde{\mathcal{R}}_j \mathcal{Q}_j \mathcal{F}_J$. *Then, for any fixed* $\beta \in [0, \frac{1}{2})$, *we have*

$$\mathcal{F}(\tilde{\mathcal{T}}_j(\mathbf{v}, q); \mathbf{v}, q) \le C \left(\frac{h_j}{h_i}\right)^{2\beta} \mathcal{F}(\mathbf{v}, q; \mathbf{v}, q) \tag{4.10}$$

for any $(\mathbf{v}, q) \in \mathbf{W}_i \times V_i$ *with* $i \le j$.

Proof. First note that (see Lemma 4.3 of [5]) for any $\beta \in [0, \frac{1}{2})$, $\boldsymbol{\varphi} \in H^1(\Omega)^{n+1}$, and $\boldsymbol{\psi} \in H^{1+\beta}(\Omega)^{n+1}$, we have

$$\left| \int_\Omega \frac{\partial \varphi_s}{\partial x_k} \frac{\partial \psi_t}{\partial x_l}\, dx \right| \le C \left(\eta^{-1} \|\varphi_s\|_{0,\Omega} + \eta^{\frac{\beta}{1-\beta}} \|\varphi_s\|_{1,\Omega} \right) \|\psi_t\|_{1+\beta,\Omega}$$

for $1 \le s, t \le n+1$, $1 \le k, l \le n$, and any $\eta > 0$. (It is important to note that continuous piecewise linear functions are in $H^{1+\beta}(\Omega)$ so that $\|\psi_t\|_{1+\beta,\Omega} < \infty$; see [7].) Hence,

$$|\mathcal{F}(\boldsymbol{\varphi}; \boldsymbol{\psi})| \le C \left(\eta^{-1} \|\boldsymbol{\varphi}\|_{0,\Omega}^2 + \eta^{\frac{\beta}{1-\beta}} \|\boldsymbol{\varphi}\|_{1,\Omega}^2 \right)^{\frac{1}{2}} \|\boldsymbol{\psi}\|_{1+\beta,\Omega}. \tag{4.11}$$

Analogous to the proof of Lemma 4.2 in [5], we use the vector inverse inequalities

$$\begin{aligned} \|\boldsymbol{\varphi}\|_{1,\Omega} &\le C h_j^{-1} \|\boldsymbol{\varphi}\|_{0,\Omega} \quad \forall\, \boldsymbol{\varphi} \in \mathbf{W}_j \times V_j, \\ \|\boldsymbol{\psi}\|_{1+\beta,\Omega} &\le C h_i^{-\beta} \|\boldsymbol{\psi}\|_{1,\Omega} \quad \forall\, \boldsymbol{\psi} \in \mathbf{W}_i \times V_i, \end{aligned} \tag{4.12}$$

for any $\beta \in [0, \frac{1}{2})$, which follow immediately from the inverse inequalities for scalar functions; see [7]. From the definitions of $\mathcal{F}_j$ and $\tilde{\mathcal{T}}_j$, we have

$$\mathcal{F}(\tilde{\mathcal{T}}_j(\mathbf{v}, q); \mathbf{v}, q) = \frac{1}{\lambda_j} \mathcal{F}(\mathcal{F}_j(\mathbf{v}, q); \mathbf{v}, q) = \frac{1}{\lambda_j} (\mathcal{F}_j(\mathbf{v}, q); \mathcal{F}_j(\mathbf{v}, q)). \tag{4.13}$$

For $(\mathbf{v}, q) \in \mathbf{W}_i \times V_i$ with $i \le j$, (4.11) and (4.12) imply

$$\|\mathcal{F}_j(\mathbf{v}, q)\|_{0,\Omega} = \sup_{(\mathbf{w}, p) \in \mathbf{W}_j \times V_j} \frac{|((\mathbf{w}, p); \mathcal{F}_j(\mathbf{v}, q))|}{\|(\mathbf{w}, p)\|_{0,\Omega}} = \sup_{(\mathbf{w}, p) \in \mathbf{W}_j \times V_j} \frac{|\mathcal{F}((\mathbf{w}, p); (\mathbf{v}, q))|}{\|(\mathbf{w}, p)\|_{0,\Omega}}$$

$$\le C(\eta^{-1} + C\eta^{\frac{\beta}{1-\beta}} h_j^{-2})^{\frac{1}{2}} h_i^{-\beta} \|(\mathbf{v}, q)\|_{1,\Omega}. \tag{4.14}$$

Setting $\eta = h_j^{2-2\beta}$ and using (4.14) in (4.13) yields (4.10). ∎

Assume that the smoothing operator $\mathcal{R}_j$ satisfies the following conditions for $j = 0, 1, \ldots, J$ and for all $(\mathbf{v}, q) \in \mathbf{W}_h \times V_h$, where $\mathcal{K}_j = I - \mathcal{R}_j \mathcal{F}_j$ and $\omega \in (0, 2)$:

$$\begin{aligned} &\frac{\|(\mathbf{v}, q)\|_{0,\Omega}^2}{\lambda_j} \le C\, ((I - \mathcal{K}_j^* \mathcal{K}_j) \mathcal{F}_j^{-1}(\mathbf{v}, q); \mathbf{v}, q), \\ &\mathcal{F}(\mathcal{T}_j(\mathbf{v}, q); \mathcal{T}_j(\mathbf{v}, q)) \le \omega \mathcal{F}(\mathcal{T}_j(\mathbf{v}, q); \mathbf{v}, q). \end{aligned}$$

These assumptions, which correspond to $A4$ and $A5$ of [5], are easily verified in general for standard smoothers like Jacobi and Gauss–Seidel (cf. [26]).

THEOREM 4.2 (Multiplicative multigrid algorithm). *We have*

$$\mathcal{F}(\mathcal{E}_J(\mathbf{v},\, q);\, \mathcal{E}_J(\mathbf{v},\, q)) \le \gamma\, \mathcal{F}(\mathbf{v},\, q;\, \mathbf{v},\, q) \quad \forall\ (\mathbf{v},\, q) \in \mathbf{W}_h \times V_h, \tag{4.15}$$

where

$$\gamma = 1 - \frac{1}{C} \in (0,\, 1).$$

Proof. Since the boundary of the domain Ω is $C^{1,\,1}$ or a convex polyhedron, it is easy to show (cf. [28, 29, 5]) that, for any $(\mathbf{v},\, q) \in \mathbf{W}_h \times V_h$,

$$(\mathbf{v},\, q;\, \mathbf{v},\, q)_{1,\,\Omega} \le C \sum_{j=0}^{J} (\tilde{\mathcal{T}}_j(\mathbf{v},\, q);\, \mathbf{v},\, q)_{1,\,\Omega}.$$

Together with Lemma 3.1 in [5] and (4.8), this inequality implies that

$$\mathcal{F}(\mathbf{v},\, q;\, \mathbf{v},\, q) \le C \sum_{j=0}^{J} \mathcal{F}(\tilde{\mathcal{T}}_j(\mathbf{v},\, q);\, \mathbf{v},\, q).$$

The theorem now follows from Theorem 3.2 in [5] and Lemma 4.1. ∎

Remark 4.2. *Similar results can be obtained for domain decomposition* (*see* [12]) *and local mesh refinement methods* (*see* [5]).

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