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FOSLL* METHOD FOR THE EDDY CURRENT PROBLEM WITH THREE-DIMENSIONAL EDGE SINGULARITIES*

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Abstract. In the case that the domain has reentrant edges, the standard finite element method loses its global accuracy because of singularities on the boundary. To overcome this difficulty, FOSLL* is applied in this paper. FOSLL* is a methodology for solving PDEs using the dual operator. Here, a modified FOSLL* method is developed that employs a partially weighted functional and allows the use of a standard finite element scheme without losing global accuracy.

Key words. eddy current problem, singularities, least squares, finite element methods

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1. Introduction.

The Maxwell equations are a set of fundamental equations governing all macroscopic electromagnetic phenomena. It is known that the numerical resolution of the full system of the Maxwell equations can be very expensive. However, it is possible to use a simplified model that approximates the Maxwell equations and explains particular problems encountered in electromagnetism. In many cases, one can use the so-called eddy current model, which is obtained by neglecting the displacement current in the Maxwell equations. Here, we consider the following two basic laws of electricity and magnetism, which form the eddy current model:

$$\text{Faraday's Law}: \quad \frac{\partial \mu \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0},$$
$$\text{Ampère's Law}: \quad \nabla \times \mathbf{H} - \sigma \mathbf{E} = \mathbf{0},$$

where $\mathbf{E}$ is the electric field intensity, $\mathbf{H}$ is the magnetic field intensity, μ is the permeability, and σ is the conductivity. We consider two types of boundary conditions

$$\mathbf{n} \times \mathbf{E} = \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{H} = 0, \qquad \text{and} \qquad \mathbf{n} \cdot \mathbf{E} = 0, \quad \mathbf{n} \times \mathbf{H} = \mathbf{0},$$

where $\mathbf{n}$ is the unit external normal vector. The electric and magnetic field intensities, $\mathbf{E}$ and $\mathbf{H}$, which follow Faraday's and Ampère's laws with homogeneous boundary conditions, satisfy

$$\mathbf{E} \in H_0(\nabla\times) \cap H(\nabla \cdot \sigma), \quad \mathbf{H} \in H(\nabla\times) \cap H_0(\nabla \cdot \mu)$$

or

$$\mathbf{E} \in H(\nabla\times) \cap H_0(\nabla \cdot \sigma), \quad \mathbf{H} \in H_0(\nabla\times) \cap H(\nabla \cdot \mu).$$

For a precise definition of the above Sobolev spaces, see section 2. In addition, if

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- μ and σ are smooth,
- either the domain is a convex polyhedron or the boundary is $\mathcal{C}^{1,1}$, and
- different types of boundary conditions do not meet at an edge with the internal angle $> \pi/2$,

then $\mathbf{E} \in (H^1)^3$ and $\mathbf{H} \in (H^1)^3$. Standard numerical techniques can be used to approximately solve the equations under the above smoothness assumptions. For example, first-order system least squares (FOSLS) with H^1-finite element spaces and multigrid methods can be used to solve these equations efficiently (cf. [3], [4], [14]). The FOSLS method is based on minimization of the squared residual norm, $||L\mathbf{V} - \mathbf{F}||_0^2$, of the system $L\mathbf{U} = \mathbf{F}$, where L represents a system of linear first-order equations, $\mathbf{U}$ a vector of unknowns, and $\mathbf{F}$ a vector of known functions. The standard least squares method approximates unknown $\mathbf{U}$ in the given H^1-finite element space when the bilinear form corresponding to $||L\mathbf{V} - \mathbf{F}||_0^2$ is equivalent to the product H^1-norm, and this H^1-equivalence is provided under sufficient smoothness assumptions on the domain, coefficients, and data of the original problem.

In the presence of discontinuous coefficients, nonsmooth, nonconvex domain, or certain irregular boundary conditions, the solution may not be in H^1. This precludes the use of H^1-conforming finite element spaces in least squares and Galerkin formulations of the Maxwell equations.

A partial list of the remedies for this loss of H^1-regularity in FOSLS can be found in [1], [5], [18], and [24]. In [5], the first-order system LL* (FOSLL*) method was introduced to overcome the difficulty that arises from discontinuous coefficients. The basic idea of the FOSLL* method can be explained by looking at a linear system of equations, $A\mathbf{x} = \mathbf{b}$. The least squares method minimizes $||A\mathbf{x} - \mathbf{b}||_0^2$, which leads to the normal equations $A^tA\mathbf{x} = A^t\mathbf{b}$. The dual of this method involves the system, $AA^t\mathbf{y} = \mathbf{b}$, where $\mathbf{x} = A^t\mathbf{y}$. FOSLL* solves $AA^t\mathbf{y} = \mathbf{b}$ by minimizing the functional $\langle A^t\mathbf{y}, A^t\mathbf{y}\rangle - 2\langle \mathbf{y}, \mathbf{b}\rangle$ which is equivalent to minimizing $||A^t\mathbf{y} - \mathbf{x}||_0^2$. For a given first-order linear system of PDEs, $L\mathbf{U} = \mathbf{F}$, the FOSLL* method solves the system, $LL^*\mathbf{U}^* = \mathbf{F}$, by minimizing the functional, $||L^*\mathbf{U}^* - \mathbf{U}||_0^2$, with the dual variable, $\mathbf{U}^*$, and the L^2-adjoint operator, L^*, of L. Minimizing $||L^*\mathbf{U}^* - \mathbf{U}||_0^2$ over $\mathbf{U}^*$ in the domain of L^* is accomplished by solving the weak problem of finding $\mathbf{U}^*$ such that

$$\langle\, L^*\mathbf{U}^*, L^*\mathbf{V}\,\rangle \;=\; \langle\, \mathbf{U}, L^*\mathbf{V}\,\rangle \;=\; \langle\, L\mathbf{U}, \mathbf{V}\,\rangle \;=\; \langle\, \mathbf{F}, \mathbf{V}\,\rangle \tag{1.1}$$

for every $\mathbf{V}$ in the domain of L^*. Then, the solution we seek is $\mathbf{U} = L^*\mathbf{U}^*$. The equation in (1.1) shows that we can solve the dual problem with the given data (right-hand side) of the original problem without knowing the exact solution, $\mathbf{U}$.

In [18], a modified FOSLL* method was developed that allows an accurate approximation using H^1-conforming finite elements for the equations having singular boundary points in two dimension. The results in [24] established a modification of the FOSLS method for the problem in a two-dimensional nonconvex domain having irregular boundary conditions. A weighted norm was used in [24] in order to reduce the difficulties from dealing with the absence of the smoothness of the problem. As a different type of remedy, one of the most common approaches is to use Raviart–Thomas or Nédélec edge elements as a finite element space [20]. These Raviart–Thomas and Nédélec edge element spaces are in $H(\nabla\cdot)$ and $H(\nabla\times)$, respectively, but not in $(H^1)^3$. Another potential form to reduce the difficulties from low regularity of the solution was introduced in [2]. The analysis in [2] is based on a weak variational formulation; the authors employ an H^{-1}-norm least-squares approach in discrete space to avoid dealing with the inf-sup condition. In [10], weighted regularization of time harmonic Maxwell equations in a polyhedral domain using a Galerkin formulation was investigated. Introducing special weights inside the divergence integral allows the approximation of nonsmooth solutions by an H^1-conforming finite element. Error estimates under the assumptions of special finite element spaces were established in [10].

As mentioned above, modifications to FOSLL* were developed that effectively handle discontinuous coefficients in two and three dimensions and irregular boundary points in two dimensions. However, there has not been any previous attempt to use FOSLL* to handle the difficulty from reentrant edges in three dimensions. First, we use standard FOSLL* to abate the difficulties from discontinuous coefficients, and then modify it to deal with the reentrant edges. We develop a modified FOSLL* using partially weighted norms in the functional to be minimized, so that we can use H^1-conforming finite elements. We do not consider the case that different types of boundary conditions meet at an edge with an internal angle greater than $\pi/2$ or the case in which the domain has conical points and vertices, where several reentrant edges meet. However, we believe that the approach developed here can be easily extended to those cases.

The approximate solution that the FOSLL* approach produces is of the form L^*U^h, where U^h is an H^1-conforming finite element. This approximation contains the curl-free Nédélec edge elements. Our approach involves a substantial decrease in computational cost over the curl-curl formulation because it is easy to implement and the resulting linear systems are easily solved by algebraic multigrid methods [23] even with higher order elements. We obtain the same error estimates as the Nédélec element approach in the L^2-norm and we can easily extend our approach to obtain the $H(\nabla\times)$-norm, while the approach in [2] provides only an L^2-error estimate. Moreover, we obtain error estimates in $H(\nabla\cdot\mu)$- and $H(\nabla\cdot\sigma)$-norms, too.

There are similarities between the FOSLL* approach developed here and the Galerkin formulation with the weighted regularization presented in [10]. While FOSLL* differs in many respects from a Galerkin formulation, under special circumstances we show that they are equivalent (see section 5). In [10], σ was assumed to be constant and E was approximated. It is easy to see that if μ were assumed constant, the same approach could be used to approximate for H. FOSLL* allows both σ and μ to be discontinuous in a natural way. We obtain the same error estimates as the approach in [10] while employing any standard H^1-conforming finite element spaces.

In this paper, we consider the Maxwell equations with discontinuous coefficients and irregular boundary. The error estimates established here hold for standard H^1-conforming finite element spaces and provide convergence rates that depend on the power of the weighting used. Numerical tests show surprising agreement with the theory. The model problem is given in section 2. In section 3, we introduce the FOSLS and FOSLL* methods briefly and explain the difficulties arising from singularities. In section 4, we modify standard FOSLL* and show that H^1-conforming elements can be used. A scaling is introduced and the connection to the Galerkin formulation in a special case is explored in section 5. In section 6, the discretization error estimates are obtained. The numerical results are given in section 7.

2. Model problem. Let Q be a polygon in $\mathbb{R}^2$ with a reentrant corner, that is, a corner that has inner angle bigger than π. Let $I \in \mathbb{R}$ be a bounded interval, and consider the prototype domain, $\Omega := Q \times I \subset \mathbb{R}^3$, which is a polyhedral cylinder. In this paper, we restrict ourselves to the case where the domain has one reentrant edge; however, the general case follows easily. By translation and rotation, we may suppose that the reentrant edge on the boundary that induces the singularity is on the z-axis.

Throughout this paper, we use $\langle\cdot,\cdot\rangle$ and $||\cdot||$ to denote the L^2-inner product and norm, respectively. We use $||\cdot||_k$ to denote the standard Sobolev H^k-norm and $|\cdot|_k$ to denote the seminorm in $H^k(\Omega)$. Let $b \in L^\infty(\Omega)$ be a scalar function, and define

$$H_0(\nabla\times) \cap H(\nabla\cdot b) := \{\mathbf{u} \in L^2(\Omega)^3 \mid ||\nabla\times \mathbf{u}||^2 + ||\nabla\cdot b\mathbf{u}||^2 < \infty,\ \mathbf{n}\times\mathbf{u} = \mathbf{0} \text{ on } \partial\Omega\},$$

$$H(\nabla\times) \cap H_0(\nabla\cdot b) := \{\mathbf{u} \in L^2(\Omega)^3 \mid ||\nabla\times \mathbf{u}||^2 + ||\nabla\cdot b\mathbf{u}||^2 < \infty,\ \mathbf{n}\cdot\mathbf{u} = 0 \text{ on } \partial\Omega\}.$$

Define $H^k_\beta(\Omega)$ as the weighted Sobolev space of functions u such that

$$||u||^2_{k,\beta} = \sum_{|m|=0}^{k} \int_\Omega r^{2(\beta+|m|-k)} \left|D^m u\right|^2 d\Omega < \infty,$$

where $r := r(\mathbf{x})$ is the distance of $\mathbf{x} \in \Omega$ from the reentrant edge. We define partially weighted norms to use in our modification of the FOSLL* functional, for $\mathbf{u}, \mathbf{v} \in L^2(\Omega)^3$ and $p, q \in H^0_\beta(\Omega)$, as

$$||(\mathbf{u}^t, p)^t||^2_\beta := ||\mathbf{u}||^2 + ||p||^2_{0,\beta}, \tag{2.1}$$

$$||(\mathbf{u}^t, p, \mathbf{v}^t, q)^t||^2_\beta := ||\mathbf{u}||^2 + ||p||^2_{0,\beta} + ||\mathbf{v}||^2 + ||q||^2_{0,\beta}. \tag{2.2}$$

In the above, note that only the scalar terms, p and q, involve weighted norms.

Now, consider the following eddy current problem:

$$\begin{aligned} \frac{\partial \mu \mathbf{H}}{\partial t} + \nabla\times\mathbf{E} &= \mathbf{0} \quad \text{in} \quad \Omega, \\ \nabla\times\mathbf{H} - \sigma\mathbf{E} &= \mathbf{0} \quad \text{in} \quad \Omega, \end{aligned} \tag{2.3}$$

with $\mathbf{E}(\mathbf{x},t)$ the electric field intensity, $\mathbf{H}(\mathbf{x},t)$ the magnetic field intensity, $\mu(\mathbf{x})$ the permeability, and $\sigma(\mathbf{x})$ the conductivity. We assume that coefficients $\mu(\mathbf{x})$ and $\sigma(\mathbf{x})$ are piecewise smooth, positive real valued, and bounded; that is, they satisfy

$$\mu_0 \le \mu(\mathbf{x}) \le \mu_1, \quad \sigma_0 \le \sigma(\mathbf{x}) \le \sigma_1 \quad \text{for all } \mathbf{x} \in \overline{\Omega}, \tag{2.4}$$

for positive constants μ_0, μ_1, σ_0, and σ_1. We consider two types of boundary conditions,

$$\begin{aligned} \text{type I}: &\quad \mathbf{n}\times\mathbf{E} = \mathbf{0}, \quad \mathbf{n}\cdot\mathbf{H} = 0, \\ \text{type II}: &\quad \mathbf{n}\cdot\mathbf{E} = 0, \quad \mathbf{n}\times\mathbf{H} = \mathbf{0}. \end{aligned}$$

Type I corresponds to perfectly conducting walls, while type II corresponds to perfectly insulating walls. Using the backward Euler approximation in time gives

$$\frac{\mu}{\delta t}\mathbf{H} + \nabla\times\mathbf{E} = \frac{\mu}{\delta t}\mathbf{H}_{\text{old}},$$

where $\mathbf{H}_{\text{old}}$ is the solution at the previous time step. Equation (2.3) implies

$$\nabla\cdot\sigma\mathbf{E} = 0, \qquad \nabla\cdot\mu\mathbf{H} = \nabla\cdot\mu\mathbf{H}_{\text{old}}. \tag{2.5}$$

Without loss of generality, we assume $\nabla \cdot \mu \mathbf{H}_{\mathrm{old}} = 0$. The resulting system then is

$$
\begin{aligned}
-\sigma \mathbf{E} + \nabla \times \mathbf{H} &= \mathbf{0}, \quad \nabla \times \mathbf{E} + \tilde{\mu} \mathbf{H} = \tilde{\mu} \mathbf{H}_{\mathrm{old}}, \\
\nabla \cdot \sigma \mathbf{E} &= 0, \quad \nabla \cdot \mu \mathbf{H} = 0,
\end{aligned} \tag{2.6}
$$

where $\tilde{\mu} = \mu/\delta t$. Since δt is a constant, $\nabla \cdot \tilde{\mu}\mathbf{H} = 0$. Let δt^{-1} be absorbed into μ and $\tilde{\mu}$ be replaced with μ. It is known that there exists a solution, $(\mathbf{E}, \mathbf{H})$, of the system (2.6) in $(H(\nabla\times) \cap H(\nabla \cdot \sigma)) \times (H(\nabla\times) \cap H(\nabla \cdot \mu))$ satisfying type I or II boundary conditions (cf. [14]). From now on, we consider only the case that the domain is surrounded by perfectly conducting walls, since the procedure is the same for perfectly insulating walls. Moreover, the case of mixed boundary conditions can be handled in a similar fashion.

In this paper, c is a generic term that is used to denote various constants. Its dependence on other quantities is indicated when necessary. For convenience of notation, superscript t for the vector transpose is omitted.

3. FOSLS and FOSLL*. In this section, we give a brief introduction to FOSLS and FOSLL* to explain the basic ideas and to show how they suffer in the presence of singularities. First, we introduce slack variables. Even though system (2.6) can be solved by itself, we extend the system since the extended system provides H^1-equivalence to the bilinear form of $||L^*\mathbf{U}^* - \mathbf{U}||$ in FOSLL* under sufficient smoothness assumptions. We extend system (2.6) by adding slack variables, s and k, to yield

$$
\begin{array}{rcccccl}
-\sigma\mathbf{E} & & & + & \nabla \times \mathbf{H} & - \nabla k & = \mathbf{0} \quad \text{in } \Omega, \\
 & - & a_1 s & + & \nabla \cdot \mu \mathbf{H} & & = 0 \quad \text{in } \Omega, \\
\nabla \times \mathbf{E} & - & \nabla s & + & \mu \mathbf{H} & & = \mu \mathbf{H}_{\mathrm{old}} \quad \text{in } \Omega, \\
\nabla \cdot \sigma \mathbf{E} & & & & & + a_2 k & = 0 \quad \text{in } \Omega,
\end{array}
$$

$$
\mathbf{n} \times \mathbf{E} = \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{H} = 0, \quad k = 0 \quad \text{on } \partial\Omega,
$$

with nonnegative constants a_1 and a_2. The above system can be rewritten as

$$
\mathcal{L}\mathbf{U} = \mathcal{L}(\mathbf{E}, s, \mathbf{H}, k) = \mathbf{F} \quad \text{in } \Omega,
$$

where

$$
\mathcal{L}\mathbf{U} = \begin{bmatrix} -\sigma I & 0 & \nabla\times & -\nabla \\ 0 & -a_1 & \nabla \cdot \mu & 0 \\ \nabla\times & -\nabla & \mu I & 0 \\ \nabla \cdot \sigma & 0 & 0 & a_2 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ s \\ \mathbf{H} \\ k \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ \mu \mathbf{H}_{\mathrm{old}} \\ 0 \end{bmatrix} = \mathbf{F}. \tag{3.1}
$$

The domain of $\mathcal{L}$ is

$$
D(\mathcal{L}) = (H_0(\nabla\times) \cap H(\nabla \cdot \sigma)) \times H^1(\Omega)/\mathbb{R} \times (H(\nabla\times) \cap H_0(\nabla \cdot \mu)) \times H_0^1(\Omega),
$$

which is a Hilbert space under the norm

$$
\begin{aligned}
||(\mathbf{E}, s, \mathbf{H}, k)||_{\mathcal{L}}^2 := {} & ||\mathbf{E}||^2 + ||\nabla \times \mathbf{E}||^2 + ||\nabla \cdot \sigma \mathbf{E}||^2 + ||s||_1^2 \\
& + ||\mathbf{H}||^2 + ||\nabla \times \mathbf{H}||^2 + ||\nabla \cdot \mu \mathbf{H}||^2 + ||k||_1^2.
\end{aligned} \tag{3.2}
$$

The range of $\mathcal{L}$ is $L^2(\Omega)^8$. It is easily shown that $s = 0$ and $k = 0$ if $(\mathbf{E}, s, \mathbf{H}, k)$ is the solution of (3.1) in $D(\mathcal{L})$ as long as the constants a_1 and a_2 are nonnegative. The FOSLS method minimizes the least-squares functional

$$\mathcal{F}(\mathbf{U}; \mathbf{F}) = ||\ \mathcal{L}\mathbf{U} - \mathbf{F}\ ||^2$$

in the weak sense, that is, we look for the solution of the corresponding weak form as follows: Find $\mathbf{U} \in D(\mathcal{L})$ satisfying

$$\langle\ \mathcal{L}\mathbf{U}, \mathcal{L}\mathbf{V}\ \rangle \ = \ \langle\ \mathbf{F}, \mathcal{L}\mathbf{V}\ \rangle \qquad \text{for all} \ \ \mathbf{V} \in D(\mathcal{L}). \tag{3.3}$$

The FOSLL* approach solves the corresponding dual problem

$$\mathcal{L}^*\mathbf{U}^* = \mathcal{L}^*(\mathcal{U}, p, \mathcal{V}, q) = \mathbf{U} \quad \text{ in } \Omega, \tag{3.4}$$

where the L^2-adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ is defined by

$$\mathcal{L}^*\mathbf{U}^* = \begin{bmatrix} -\sigma I & 0 & \nabla\times & -\sigma\nabla \\ 0 & -a_1 & \nabla\cdot & 0 \\ \nabla\times & -\mu\nabla & \mu I & 0 \\ \nabla\cdot & 0 & 0 & a_2 \end{bmatrix} \begin{bmatrix} \mathcal{U} \\ p \\ \mathcal{V} \\ q \end{bmatrix} = \begin{bmatrix} \mathbf{E} \\ s \\ \mathbf{H} \\ k \end{bmatrix} = \mathbf{U}, \tag{3.5}$$

and $\mathcal{L}^* : D(\mathcal{L}^*) \to L^2(\Omega)^8$ with

$$D(\mathcal{L}^*) = (H_0(\nabla\times) \cap H(\nabla\cdot)) \times H^1(\Omega)/\mathbb{R} \times (H(\nabla\times) \cap H_0(\nabla\cdot)) \times H_0^1(\Omega).$$

To solve the dual problem we minimize the dual functional

$$\mathcal{F}^*(\mathbf{U}^*; \mathbf{U}) = ||\ \mathcal{L}^*\mathbf{U}^* - \mathbf{U}\ ||^2 \tag{3.6}$$

on $D(\mathcal{L}^*)$. The corresponding weak form is the following: Find $\mathbf{U}^* \in D(\mathcal{L}^*)$ satisfying

$$\langle\ \mathcal{L}^*\mathbf{U}^*, \mathcal{L}^*\mathbf{V}^*\ \rangle = \langle\ \mathbf{U}, \mathcal{L}^*\mathbf{V}^*\ \rangle = \langle\ \mathbf{F}, \mathbf{V}^*\ \rangle \qquad \text{for all} \ \ \mathbf{V}^* \in D(\mathcal{L}^*), \tag{3.7}$$

where $\mathbf{U}$ is the solution of (3.1) for given $\mathbf{F}$. Equation (3.7) shows that we can solve the dual problem with the given data, $\mathbf{F}$, of the original problem without knowing the solution, $\mathbf{U}$. Then, we obtain the solution from (3.4), $\mathbf{U} = \mathcal{L}^*\mathbf{U}^*$.

LEMMA 3.1. *There exists a unique solution,* $\mathbf{U} \in D(\mathcal{L})$*, satisfying* (3.3).

Proof. Let $(E, e, H, h) \in D(\mathcal{L})$. Using the same manner which was used to prove Lemmas 3.4 and 3.6 in [12] for E, H and the Poincaré inequality for e, h, we have

$$||(E, e, H, h)||_{\mathcal{L}}^2 \leq c\left(||\nabla\times E||^2 + ||\nabla\cdot \sigma E||^2 + |e|_1^2 + ||\nabla\times H||^2 + ||\nabla\cdot \mu H||^2 + |h|_1^2\right).$$

Since $\mathbf{n} \times E = \mathbf{0}$ and $h = 0$ on the boundary, the conditions in (2.4) provide

$$\begin{aligned} \mu_1^{-1}||\nabla\times E||^2 \ &\leq \ \left\langle \mu^{-1}\nabla\times E, \nabla\times E - \nabla e + \mu H\right\rangle - \langle\nabla\times E, H\rangle, \\ {\sigma_1}^{-1}||\nabla\times H||^2 \ &\leq \ \left\langle \sigma^{-1}\nabla\times H, -\sigma E + \nabla\times H - \nabla h\right\rangle + \langle\nabla\times H, E\rangle. \end{aligned}$$

The above two inequalities, together with Hölder's inequality and the ϵ-inequality, give

$$||\nabla\times E||^2 + ||\nabla\times H||^2 \leq c\left(||\nabla\times E - \nabla e + \mu H||^2 + ||-\sigma E + \nabla\times H - \nabla h||^2\right).$$

Consider the following several different cases for a_1 and a_2:

(a) If $a_1 \neq 0$ and $a_2 \neq 0$, then, by Green's formula,

$$||\nabla \cdot \sigma E||^2 = \langle \nabla \cdot \sigma E, \nabla \cdot \sigma E + a_2 h \rangle + a_2 \langle \sigma E, \nabla h \rangle , \tag{3.8}$$

$$||\nabla \cdot \mu H||^2 = \langle \nabla \cdot \mu H, \nabla \cdot \mu H - a_1 e \rangle - a_1 \langle \mu H, \nabla e \rangle , \tag{3.9}$$

$$||\nabla e||^2 = \langle \nabla e, -\nabla \times E + \nabla e - \mu H \rangle + \langle \nabla e, \mu H \rangle , \tag{3.10}$$

$$||\nabla h||^2 = \langle \nabla h, \sigma E - \nabla \times H + \nabla h \rangle - \langle \nabla h, \sigma E \rangle . \tag{3.11}$$

Multiply (3.10) by a_1 and (3.11) by a_2 and add to (3.9) and (3.8), respectively. Again use Hölder's inequality and the ϵ-inequality to obtain

$$||\nabla \cdot \sigma E||^2 + ||\nabla \cdot \mu H||^2 + ||\nabla e||^2 + ||\nabla h||^2 \leq c \, ||\mathcal{L}(E, e, H, h)||^2.$$

(b) If $a_1 = a_2 = 0$, taking Hölder's and Poincaré inequalities in (3.10) and (3.11) implies $||\nabla e||^2 + ||\nabla h||^2 \leq c \, ||\mathcal{L}(E, e, H, h)||^2$.

(c) If only one of a_1 and a_2 is 0, for example $a_1 = 0$ and $a_2 \neq 0$, then we use the same calculation in case (a) for ∇e. Multiply (3.11) by a_2 and add it to (3.8) to get $||\nabla \cdot \sigma E||^2 + ||\nabla h||^2 \leq c \, (||\nabla \cdot \sigma E + a_2 h||^2 + || - \sigma E + \nabla \times H - \nabla h||^2)$.

Thus, $||(E, e, H, h)||^2_{\mathcal{L}} \leq c \, ||\mathcal{L}(E, e, H, h)||^2$, so that $\mathcal{L}$ is coercive. It is easy to prove the continuity of $\mathcal{L}$ by using the triangle inequality. Therefore, by the Lax–Milgram theorem, there exists the solution of (3.3). □

Now, we consider the dual weak problem (3.7). In a similar manner, we can show the existence and uniqueness of the solution for (3.7).

LEMMA 3.2. *There exists a unique solution,* $\mathbf{U}^* \in D(\mathcal{L}^*)$*, satisfying* (3.7).

COROLLARY 3.3. *The operator* $\mathcal{L} : D(\mathcal{L}) \to L^2(\Omega)^8$*, defined in* (3.1)*, is bijective.*

Proof. In Lemmas 3.1 and 3.2, it is proved that $\mathcal{L}$ and $\mathcal{L}^*$, defined in (3.1) and (3.5), respectively, are coercive. Therefore, $\mathcal{L}$ and $\mathcal{L}^*$ are injective. The coercivity and continuity of $\mathcal{L}$ provide that $\mathcal{L}$ is a closed operator. Then, by the closed range theorem (cf. [25]), the injectivity of $\mathcal{L}^*$ induces the surjectivity of $\mathcal{L}$. Thus, $\mathcal{L}$ is bijective. □

COROLLARY 3.4. *The operator* $\mathcal{L}^* \colon D(\mathcal{L}^*) \to L^2(\Omega)^8$*, defined in* (3.5)*, is bijective.*

Proof. Since $\mathcal{L}$ is a closed operator, by Lemma 2.1 in [5] and Corollary 3.3, $\mathcal{L}^*$ is bijective. □

Remark 3.5. Corollary 3.4 implies that, for given $\mathbf{F} \in L^2(\Omega)^8$, there exists a unique solution $\mathbf{U}^* \in D(\mathcal{L}^*)$ satisfying the weak form (3.7).

We consider several cases that incur difficulties in approximately solving the eddy current problem with H^1-conforming finite elements. Suppose that there are no boundary singularities but μ and σ are not smooth. Because the coefficients are not smooth, $D(\mathcal{L})$ is not imbedded into $H^1(\Omega)^8$. In fact, $H^1(\Omega)^8$ is a closed, proper subspace of $D(\mathcal{L})$. Therefore, H^1-conforming finite element spaces cannot be used to approximate the solution of system (3.1). The FOSLL* method may be used to overcome this difficulty. The efficiency of FOSLL* in this context can be seen by observing the dual operator $\mathcal{L}^*$ in (3.5). All of the discontinuous coefficients inside the derivatives in the $\mathcal{L}$ system are outside the differential operators in the $\mathcal{L}^*$ system. Accordingly, we have $D(\mathcal{L}^*)$ imbedded into $H^1(\Omega)^8$. Now, we suppose that μ and σ are not smooth and there is a boundary singularity. Although we can resolve the difficulty with the discontinuous coefficients by applying the standard FOSLL* method, the boundary singularity still leads to

$$H_0(\nabla\times) \cap H(\nabla\cdot) \not\subset H^1(\Omega)^3 \quad \text{and} \quad H(\nabla\times) \cap H_0(\nabla\cdot) \not\subset H^1(\Omega)^3.$$

In [18], a modification of the FOSLL* method was developed that overcomes this difficulty for the general scalar elliptic PDEs in the plane. In this paper, we introduce a different type of modification of FOSLL* to mitigate the difficulties with boundary singularities in three space dimensions.

4. The modified FOSLL* method. In this section, we present a modified FOSLL* functional in which the second and fourth equations in (3.5) involve weighted norms, that is, the functional is given by $\|\mathcal{L}^*\mathbf{U}^* - \mathbf{U}\|_\alpha^2$. Note that we have used the partially weighted norm that was introduced in (2.2). In subsection 4.2, we show how this modified FOSLL* functional works in the presence of singularities. Before getting into the details about the modified FOSLL* functional, we first show several Poincaré-type inequalities which are useful in many places. The first lemma appears in [15].

LEMMA 4.1. *Let $\Omega = \Omega_1 \times (a,b)$ with $\Omega_1 = \{(r,\theta) | 0 < r < R < 1, 0 < \theta < \omega, 0 < \omega \le 2\pi\}$. If $q \in H^1_{\beta+1}(\Omega)$ vanishes on $\partial\Omega$, then, for any β,*

$$||q||_{0,\beta} \quad \le \quad c\ ||\nabla q||_{0,\beta+1}. \tag{4.1}$$

Using the above lemma, we show the following.

LEMMA 4.2. *Assume that Ω is bounded, Lipschitz continuous, and simply connected. Let $\phi \in H_0(\nabla\times) \cap H(\nabla\cdot)$; then there exists a constant c such that, for any $0 \le \alpha \le 1$,*

$$||\phi|| \ \le \ c\ \left(||\nabla\times\phi|| + ||r^\alpha \nabla\cdot\phi||\right).$$

Proof. Let $\phi \in H_0(\nabla\times) \cap H(\nabla\cdot)$. By Lemma 3.4 in [12], ϕ can be written as

$$\phi = \varphi + \nabla\xi, \tag{4.2}$$

where $\varphi \in H_0(\nabla\times) \cap H(\nabla\cdot)$, $\nabla\cdot\varphi = 0$, and $\xi \in H^1_0(\Omega)$ satisfies $\Delta\xi = \nabla\cdot\phi$. Using the Cauchy–Schwarz inequality, Lemma 4.1, and the assumption on α yields

$$\begin{aligned} ||\nabla\xi||^2 = \langle\nabla\xi,\nabla\xi\rangle \ &= \ \langle -\nabla\cdot\phi, \xi\rangle \ \le \ ||r^\alpha\nabla\cdot\phi||\ ||r^{-\alpha}\xi|| \\ &\le c||r^\alpha\nabla\cdot\phi||\ ||r^{1-\alpha}\nabla\xi|| \ \le \ c||r^\alpha\nabla\cdot\phi||\ ||\nabla\xi||. \end{aligned} \tag{4.3}$$

Now, (4.2), (4.3), and Lemma 3.4 in [12] imply

$$||\phi|| \le c(||\varphi|| + ||\nabla\xi||) \le c(||\nabla\times\varphi|| + ||r^\alpha\nabla\cdot\phi||) = c(||\nabla\times\phi|| + ||r^\alpha\nabla\cdot\phi||). \qquad \square$$

Lemmas 4.3 and 4.4 basically claim the same inequality in Lemma 4.1 without the zero boundary condition.

LEMMA 4.3. *Assume Ω is the same as in Lemma 4.1 and $\beta > -1$. For $p \in H^1_{\beta+1}(\Omega)$, there exists a constant c such that*

$$||p||_{0,\beta} \ \le \ c\ (||p||_{0,\beta+1} + ||\nabla p||_{0,\beta+1}).$$

Proof. Let $R_0 = \frac{R}{4}$, and let χ be a smooth function defined in Ω such that $\chi(r) = 1$ when $r < R_0$ and $\chi(r) = 0$ when $r > 2R_0$ and $|\chi'| \le cR_0^{-1}$ for some constant c. Since $1 = \chi + 1 - \chi$,

$$\int_0^R r^{2\beta}|p|^2 r dr = \int_0^R r^{2\beta}\left|\chi p + (1-\chi)p\right|^2 r dr \le 2\int_0^R r^{2\beta}\left(\left|\chi p\right|^2 + \left|(1-\chi)p\right|^2\right) r dr.$$

By the modified Hardy's inequality in [16], for $\beta > -1$,

$$\int_0^R r^{2\beta}|\chi p|^2 r\ dr \le c\int_0^R r^{2\beta+2}\left|\frac{\partial(\chi p)}{\partial r}\right|^2 r\ dr \le c\int_0^{2R_0} r^{2\beta+2}\left(\frac{1}{R_0^2}|p|^2+\left|\frac{\partial p}{\partial r}\right|^2\right) r\ dr.$$

Since $(1-\chi)p$ has nonzero values only on (R_0, R),

$$\int_0^R r^{2\beta}|(1-\chi)p|^2 r\ dr = \int_{R_0}^R r^{2\beta}|(1-\chi)p|^2 r\ dr = \int_{R_0}^R r^{-2}\ r^{2\beta+2}|(1-\chi)p|^2 r\ dr$$

$$\le {R_0}^{-2}\ \int_{R_0}^R r^{2\beta+2}|(1-\chi)p|^2 r\ dr \le {R_0}^{-2}\ \int_0^R r^{2\beta+2}|p|^2 r\ dr.$$

Hence

$$\int_\Omega r^{2\beta}|p|^2 d\Omega \le c\ R^{-2}\int_\Omega r^{2\beta+2}|p|^2 d\Omega + c\int_\Omega r^{2\beta+2}|\nabla p|^2 d\Omega. \qquad \square$$

To handle $||p||_{0,\beta+1}$ in Lemma 4.3, we prove the following lemma.

LEMMA 4.4. *Let $p \in H^1(\Omega)$ satisfying $||\nabla p||_{\beta+1-\epsilon} < \infty$; then there exist constants b and c such that, for any $\beta > -1$ and $\epsilon > 0$,*

$$||p-b||_{0,\beta}\ \le\ c\ ||\nabla p||_{0,\beta+1-\epsilon}.$$

Proof. Here, we show an outline of the proof. The details can be found in [17]. Let $p \in H^1(\Omega)$ satisfy the assumption and consider the following expression for P:

$$\begin{aligned}
&p(r,\theta,z) - p(r_0,\theta_0,z_0)\\
&= p(r,\theta,z) - p(r,\theta_0,z) + p(r,\theta_0,z) - p(r_0,\theta_0,z) + p(r_0,\theta_0,z) - p(r_0,\theta_0,z_0)\\
&= \int_{\theta_0}^{\theta}\frac{\partial p}{\partial\tilde\theta}(r,\tilde\theta,z)\ d\tilde\theta + \int_{r_0}^{r}\frac{\partial p}{\partial\tilde r}(\tilde r,\theta_0,z)\ d\tilde r + \int_{z_0}^{z}\frac{\partial p}{\partial\tilde z}(r_0,\theta_0,\tilde z)\ d\tilde z.
\end{aligned}$$

Multiply by $r_0^{\beta+\frac12}$ and perform the integration $\int_\Omega r_0 dr_0 d\theta_0 dz_0$ on both sides:

$$\begin{aligned}
c_1 p(r,\theta,z) &= \int_\Omega r_0^{\beta+\frac12} p(r_0,\theta_0,z_0) r_0 dr_0 d\theta_0 dz_0 + \int_\Omega r_0^{\beta+\frac12}\left\{\int_{\theta_0}^{\theta}\frac{\partial p}{\partial\tilde\theta}(r,\tilde\theta,z)\ d\tilde\theta\right.\\
&\quad \left. + \int_{r_0}^{r}\frac{\partial p}{\partial\tilde r}(\tilde r,\theta_0,z)\ d\tilde r + \int_{z_0}^{z}\frac{\partial p}{\partial\tilde z}(r_0,\theta_0,\tilde z)\ d\tilde z\right\} r_0 dr_0 d\theta_0 dz_0,
\end{aligned} \tag{4.4}$$

where $c_1 = \int_\Omega r_0^{\beta+\frac12} r_0 dr_0 d\theta_0 dz_0$. Let

$$b = \frac{1}{c_1}\int_\Omega r_0^{\beta+\frac12} p(r_0,\theta_0,z_0)\ r_0 dr_0 d\theta_0 dz_0;$$

then $|b| \le c||p|| < \infty$. Subtracting b from both sides in (4.4), changing the order of integration, inserting $\tilde r^{\frac{-1+\epsilon}{2}}\cdot\tilde r^{\frac{1-\epsilon}{2}} = 1$ in order to group $\tilde r^{\frac{1-\epsilon}{2}}$ with the $\frac{\partial p}{\partial\tilde r}$ term, using the Cauchy–Schwarz inequality, and squaring both sides yield

$$\begin{aligned}
|p(r,\theta,z)-b|^2 &\le c\left\{\int_0^\omega\left|\frac{\partial p}{\partial\tilde\theta}(r,\tilde\theta,z)\right|^2 d\tilde\theta + \int_0^\omega\int_0^R \tilde r^{2\beta+3}\left|\frac{\partial p}{\partial\tilde r}(\tilde r,\theta_0,z)\right|^2 d\tilde r d\theta_0\right.\\
&\left. + \int_0^\omega\left\{\frac{R^\epsilon}{\epsilon}\int_r^R \tilde r^{1-\epsilon}\left|\frac{\partial p}{\partial\tilde r}(\tilde r,\theta_0,z)\right|^2 d\tilde r + \int_0^R r_0^{2\beta+3}\int_a^b\left|\frac{\partial p}{\partial\tilde z}(r_0,\theta_0,\tilde z)\right|^2 d\tilde z dr_0\right\} d\theta_0\right\}.
\end{aligned}$$

To establish the weighted L^2-norm of $|p-b|$, multiply by $r^{2\beta}$ and take an integration over Ω. Then, we have

$$\int_\Omega r^{2\beta}|p(r,\theta,z)-b|^2 d\Omega \le c \int_\Omega r^{2\beta+2}\left(\left|\frac{1}{r}\frac{\partial p}{\partial \theta}\right|^2 + \left|\frac{\partial p}{\partial r}\right|^2 + \left|\frac{\partial p}{\partial z}\right|^2\right) + r^{2\beta+2-\epsilon}\left|\frac{\partial p}{\partial r}\right|^2 d\Omega$$

$$\le c\int_\Omega r^{2\beta+2-\epsilon}|\nabla p\,|^2 d\Omega,$$

where $c = c(\Omega,\beta,\epsilon,(\beta+1)^{-1},\epsilon^{-1}) \to \infty$ as $\epsilon \to 0$ and $\beta \to -1$. ◻

LEMMA 4.5. *Assume that Ω is bounded, Lipschitz continuous, and simply connected. Let $\psi \in H(\nabla\times)\cap H_0(\nabla\cdot)$; then there exists a constant c such that, for any $0 \le \alpha < 1$,*

$$||\psi|| \;\le\; c\;(||\nabla\times\psi|| + ||r^\alpha \nabla\cdot\psi||).$$

Proof. The proof follows similarly to Lemma 4.2 using Lemmas 4.3 and 4.4. ◻

If a vector function is in H^1 and satisfies certain boundary conditions, then the sum of norms of div and curl is equal to the semi-H^1-norm.

LEMMA 4.6. *Let Ω be a bounded polyhedral domain in $\mathbb{R}^3$. If $\mathbf{v} \in H^1(\Omega)^3$ and satisfies $\mathbf{n}\cdot\mathbf{v} = 0$ or $\mathbf{n}\times\mathbf{v} = \mathbf{0}$ on the boundary $\partial\Omega$, then*

$$||\nabla\cdot\mathbf{v}||^2 + ||\nabla\times\mathbf{v}||^2 = ||\nabla\mathbf{v}||^2.$$

Proof. See [7] and [8]. ◻

The basic idea of the modification here is to use a weighted norm in certain terms of the least squares functional in (3.6). Using a weighted norm allows the existence of a sequence, $\{\mathbf{U}_n\} \subset D(\mathcal{L}^*) \cap H^1(\Omega)^8$, converging to the nonsmooth solution, $\mathbf{U}^* \in D(\mathcal{L}^*)$, in the functional norm. Consider the operator $\mathcal{L}^*$ blockwise. Let $D_A = H_0(\nabla\times)\cap H(\nabla\cdot)$ and $D_B = H(\nabla\times)\cap H_0(\nabla\cdot)$ and define

$$A = \begin{bmatrix} \nabla\times & -\mu\nabla \\ \nabla\cdot & 0 \end{bmatrix} \quad\text{and}\quad B = \begin{bmatrix} \nabla\times & -\sigma\nabla \\ \nabla\cdot & 0 \end{bmatrix}. \tag{4.5}$$

We first show that there exist sequences $\{\mathcal{X}_n\}$ and $\{\mathcal{Y}_n\}$ in $H^1(\Omega)^4$ such that

$$||A\mathcal{X}_n - F||_\alpha \longrightarrow 0 \;\text{ and }\; ||B\mathcal{Y}_n - G||_\alpha \longrightarrow 0 \;\text{ as }\; n \longrightarrow \infty, \tag{4.6}$$

for given $F, G \in L^2(\Omega)^4$ and the norm $||\cdot||_\alpha$ defined in (2.1). We again emphasize that this notation implies that only the scalar term, the term involving $\nabla\cdot$, is weighted. Then, we discuss the density of H^1-functions in D_A and D_B under weighted norms.

4.1. The density arguments in D_A and D_B. As a first step to show the existence of H^1-sequences satisfying (4.6), we apply the well-known L^2-decomposition and show several lemmas. The next lemma provides the decomposition of $L^2(\Omega)^3$.

LEMMA 4.7. *Every function $\mathbf{w} \in L^2(\Omega)^3$ has the orthogonal decomposition*

$$\mathbf{w} = \nabla\times\mathbf{u} + \nabla\psi,$$

where $\psi \in H^1(\Omega)/\mathbb{R}$ is the only solution of $\langle\nabla\psi,\nabla\xi\rangle = \langle\mathbf{w},\nabla\xi\rangle$, for any $\xi \in H^1(\Omega)$, and $\mathbf{u} \in H^1(\Omega)^3$ satisfies $\nabla\cdot\mathbf{u} = 0$.

Proof. See [12] for details. ◻

LEMMA 4.8. *For given* $F \in L^2(\Omega)^4$*, there exists a unique solution,* $\mathcal{X} \in D_A \times H^1(\Omega)/\mathbb{R}$ *, of* $A\mathcal{X} = F$*.*

Proof. The result follows from a proof similar to the proofs of section 3. ∎

Analogously, we show the following lemma.

LEMMA 4.9. *For given* $G \in L^2(\Omega)^4$*, there exists a unique solution,* $\mathcal{Y} \in D_B \times H^1_0(\Omega)$*, of* $B\mathcal{Y} = G$*.*

Now, we provide some decompositions in D_A and D_B.

THEOREM 4.10. *Given* $\tilde{\mathbf{u}} \in D_A$*, there exists* $\mathbf{u} \in H^1(\Omega)^3 \cap D_A$ *and* $\phi \in H^1_0(\Omega)$ *such that*

$$\tilde{\mathbf{u}} = \mathbf{u} + \nabla\phi.$$

Proof. Use Lemma 4.7 to write $\nabla \times \tilde{\mathbf{u}} = \nabla \times \mathbf{u}_0 + \nabla\psi$ with $\mathbf{u}_0 \in H^1(\Omega)^3$ and $\psi \in H^1(\Omega)/\mathbb{R}$. Taking the divergence of the above equation leads to the conclusion that $\psi = 0$. Thus, $\nabla \times (\tilde{\mathbf{u}} - \mathbf{u}_0) = \mathbf{0}$, which implies that $\tilde{\mathbf{u}} = \mathbf{u}_0 + \nabla\phi_0$, for some $\phi_0 \in H^1(\Omega)/\mathbb{R}$. Now, $\mathbf{0} = \mathbf{n} \times \tilde{\mathbf{u}} = \mathbf{n} \times \mathbf{u}_0 + \mathbf{n} \times \nabla\phi_0$. Since $\mathbf{u}_0 \in H^1(\Omega)^3$, we have $\mathbf{n} \times \mathbf{u}_0 \in H^{\frac{1}{2}}(\partial\Omega)^3$. Thus, $\mathbf{n} \times \nabla\phi_0 = -\mathbf{n} \times \mathbf{u}_0$ on $\partial\Omega$, which implies $trace(\phi_0) \in H^{\frac{3}{2}}(\partial\Omega)$. Let $\phi_2 \in H^2(\Omega)$ satisfy $trace(\phi_0) = trace(\phi_2)$. Then, let

$$\mathbf{u} = \mathbf{u}_0 + \nabla\phi_2, \quad \phi = \phi_0 - \phi_2.$$

Since $\mathbf{n} \times \nabla\phi = \mathbf{0}$, the theorem is proved. ∎

THEOREM 4.11. *Given* $\tilde{\mathbf{v}} \in D_B$*, there exists* $\mathbf{v} \in H^1(\Omega)^3 \cap D_B$ *and* $\psi \in H^1(\Omega)$ *with* $\mathbf{n} \cdot \nabla\psi = 0$ *on* $\partial\Omega$ *such that*

$$\tilde{\mathbf{v}} = \mathbf{v} + \nabla\psi.$$

Proof. The proof is similar to the proof of Theorem 4.10. Here, we construct ψ satisfying $\mathbf{n} \cdot \nabla\psi = 0$ on the boundary. ∎

In the domain with a reentrant edge, the solution of the Poisson equation

$$-\Delta\phi = f$$

for $f \in L^2(\Omega)$, with a Dirichlet or Neumann boundary condition is, in general, not in $H^2(\Omega)$. It is in $H^2_{loc}(\Omega)$; that is, $\phi \in H^2(S)$ for any open subset S of Ω such that its closure $\overline{S}$ does not meet the reentrant edge (cf. [13]). The solution, ϕ, is also in $H^{1+\gamma}(\Omega)$ for some $\gamma \in (0, 1)$. A more precise measure is given by the weighted Sobolev space. This solution ϕ is in $H^2_\beta(\Omega)$ with β related to the angle of the reentrant edge (cf. [15], [21]). In the following theorems, we establish H^1-sequences satisfying (4.6).

From this point forward, if not mentioned explicitly, Ω is the prototype domain which was defined in section 2.

THEOREM 4.12. *For given* $F \in L^2(\Omega)^4$ *and an operator* A *defined in* (4.5)*, there exists a sequence* $\{\mathcal{X}_n\} \subset H^1(\Omega)^4 \cap \big(D_A \times H^1(\Omega)/\mathbb{R}\big)$ *such that*

$$\|A\mathcal{X}_n - F\|_\alpha \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

where $\alpha > 1 - \lambda$*,* $\lambda = \pi/\omega$*, and* ω *is the angle of the reentrant edge.*

Proof. Let $F = (\mathbf{f}_1, f_2) \in L^2(\Omega)^4$. From Lemma 4.8, we have $\tilde{\mathbf{u}} \in D_A$ and $\tilde{p} \in H^1(\Omega)/\mathbb{R}$ satisfying $\nabla \times \tilde{\mathbf{u}} - \mu\nabla\tilde{p} = \mathbf{f}_1$ and $\nabla \cdot \tilde{\mathbf{u}} = f_2$. By Theorem 4.10, $\tilde{\mathbf{u}}$ is decomposed of $\tilde{\mathbf{u}} = \mathbf{u} + \nabla\phi$, where $\mathbf{u} \in H^1(\Omega)^3 \cap D_A$ and $\phi \in H^1_0(\Omega)$. Here, ϕ satisfies

$$\begin{cases} \nabla \cdot \nabla\phi = -\nabla \cdot \mathbf{u} + f_2 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

Given $\alpha > 1-\lambda$, choose β such that $\beta < \alpha$ and $|\beta - 1| < \lambda$. It is known that the solution ϕ is in $H^1(\Omega) \cap H^2_\beta(\Omega)$. Define $\Omega_n = (\{(x,y)|\ 1/(2n) \le r \le 1/n\} \times \mathbb{R}) \cap \Omega$ with $r = \sqrt{x^2+y^2}$ and $\delta_n(r)$ a smooth function satisfying

$$\delta_n(r) = \begin{cases} 0 & \text{if } r < 1/(2n), \\ 1 & \text{if } r > 1/n, \end{cases} \tag{4.8}$$

where $|\ \delta_n'\ | \ \le\ c_1 n$ and $|\ \delta_n''\ | \ \le\ c_2 n^2$, for some positive constants c_1 and c_2. Define $\phi_n = \delta_n \phi$; then $\phi_n \in H^2(\Omega)$ (cf. [13]). Therefore, $\mathbf{u}_n := \mathbf{u} + \nabla \phi_n$ is in $H^1(\Omega)^3 \cap D_A$ and satisfies

$$\nabla \times \mathbf{u}_n - \mu \nabla \tilde{p} = \nabla \times \mathbf{u} - \mu \nabla \tilde{p} = \nabla \times \tilde{\mathbf{u}} - \mu \nabla \tilde{p} = \mathbf{f}_1.$$

Using the triangle inequality several times and the properties of δ_n yields

$$\begin{aligned} &\|\nabla \cdot \mathbf{u}_n - f_2\|^2_{0,\alpha} = \|\nabla \cdot (\mathbf{u} + \nabla \phi_n) - \nabla \cdot (\mathbf{u} + \nabla \phi)\|^2_{0,\alpha} = \int_\Omega r^{2\alpha} |\Delta((\delta_n(r)-1)\phi)|^2 \, d\Omega \\ &= \int\!\!\int \left(\int_0^{\frac{1}{2n}} r^{2\alpha} |\Delta \phi|^2 \ r dr + \int_{\frac{1}{2n}}^{\frac{1}{n}} r^{2\alpha} |\Delta((\delta_n(r)-1)\phi)|^2 \ r dr \right) d\theta dz \\ &\le\ c \left(\frac{1}{2n}\right)^{2(\alpha-\beta)} ||\Delta \phi||^2_{0,\beta} + c \int_{\Omega_n} r^{2\alpha} \left(|\Delta \phi|^2 + n^4 |\phi|^2 + n^2 \left(|\partial_x \phi|^2 + |\partial_y \phi|^2 \right) \right) d\Omega \\ &\le\ c\ n^{-2(\alpha-\beta)} |\phi|^2_{2,\beta} + c\ n^{-2(\alpha-\beta)} ||\phi||^2_{2,\beta} = c\ n^{-2(\alpha-\beta)} ||\phi||^2_{2,\beta}. \end{aligned}$$

The right-hand side goes to 0 as n goes to infinity. By letting $\mathcal{X}_n := (\mathbf{u}_n, \tilde{p})$, the proof is completed. ∎

We can show the next theorem in the same manner.

THEOREM 4.13. *For given $G \in L^2(\Omega)^4$ and an operator B defined in* (4.5), *there exists a sequence $\{\mathcal{Y}_n\} \subset H^1(\Omega)^4 \cap \left(D_B \times H^1_0(\Omega)\right)$ such that*

$$\|B\mathcal{Y}_n - G\|_\alpha \longrightarrow 0 \qquad \text{as} \quad n \to \infty,$$

where $\alpha > 1-\lambda$, $\lambda = \pi/\omega$, and ω is the angle of the reentrant edge.

Now, we state some density results. Define

$$D_{A_\alpha} := \{\mathbf{u} \in L^2(\Omega)^3 \mid ||\nabla \times \mathbf{u}|| + ||\nabla \cdot \mathbf{u}||_{0,\alpha} < \infty,\ \mathbf{n} \times \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega\}, \tag{4.9}$$

$$D_{B_\alpha} := \{\mathbf{u} \in L^2(\Omega)^3 \mid ||\nabla \times \mathbf{u}|| + ||\nabla \cdot \mathbf{u}||_{0,\alpha} < \infty,\ \mathbf{n} \cdot \mathbf{u} = 0 \text{ on } \partial\Omega\}, \tag{4.10}$$

which are Hilbert spaces under the norm $||\mathbf{u}||_{D_{A_\alpha}} = ||\mathbf{u}||_{D_{B_\alpha}} := (||\mathbf{u}||^2 + ||\nabla \times \mathbf{u}||^2 + ||\nabla \cdot \mathbf{u}||^2_{0,\alpha})^{\frac{1}{2}}$. The density statement for D_{A_α} can be found in [8], [10], and [11] for $\alpha \in (1-\lambda, 1)$. Here, we extend the density results to $\alpha > 1-\lambda$.

THEOREM 4.14. *$D_{A_\alpha} \cap H^1(\Omega)^3$ is dense in D_{A_α} when $\alpha > 1-\lambda$, and $D_{B_\alpha} \cap H^1(\Omega)^3$ is dense in D_{B_α} when $\alpha > 1-\lambda$.*

Proof. We separate the proof into two cases. First, we consider $1-\lambda < \alpha < 1$. Let the operator A be defined as in (4.5) and let $(\mathbf{u}, p) \in D_{A_\alpha} \times H^1(\Omega)/\mathbb{R}$; then,

$$\begin{aligned} \|A(\mathbf{u},p)\|^2_\alpha &= \|\nabla \times \mathbf{u} - \mu \nabla p\|^2 + ||\nabla \cdot \mathbf{u}||^2_{0,\alpha} \ge \mu_0 ||(1/\sqrt{\mu}) \nabla \times \mathbf{u} - \sqrt{\mu} \nabla p||^2 + ||\nabla \cdot \mathbf{u}||^2_{0,\alpha} \\ &\ge c(||\nabla \times \mathbf{u}||^2 + ||\nabla p||^2 + ||\nabla \cdot \mathbf{u}||^2_{0,\alpha}) \ge c(||\mathbf{u}||^2_{D_{A_\alpha}} + ||p||^2_1). \end{aligned}$$

In the above, Lemma 4.2 and Theorem 4.12 imply the density in D_{A_α} for $1-\lambda < \alpha < 1$.

Now, we consider the case $\alpha \geq 1$. Let $\mathbf{u} \in D_{A_\alpha}$; then, similarly to Theorem 4.10, we can show that $\mathbf{u}$ is decomposed in the form of $\mathbf{u} = \mathbf{u}_0 + \nabla\phi$, where $\mathbf{u}_0 \in H^1(\Omega)^3 \cap D_{A_\alpha}$ and $\phi \in H_0^1(\Omega)$. Let Ω_n and the smooth cut-off function $\delta_n(r)$ be defined as in the proof of Theorem 4.12, and define $\Omega_{\tilde{n}} = (\{(x,y)|r \leq 1/n\} \times \mathbb{R}) \cap \Omega$. Define $\mathbf{u}_n = \mathbf{u}_0 + \nabla(\delta_n(r)\phi)$; then $\mathbf{u}_n$ is in $H^1(\Omega)^3 \cap D_{A_\alpha}$. Since $\phi \in H_0^1(\Omega)$ and $\|\nabla\cdot\nabla\phi\|_{0,\alpha} < \infty$, it is easy to see that

$$\|\Delta\phi\|_{0,\alpha,\Omega_{\tilde{n}}} \to 0 \quad \text{and} \quad \|\phi\|_{1,\Omega_{\tilde{n}}} \to 0 \tag{4.11}$$

as $n \to \infty$, where the subscript $\Omega_{\tilde{n}}$ means the integration over $\Omega_{\tilde{n}}$. Therefore, the triangle inequality and the property of $\delta_n(r)$ yield

$$\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_n\|^2_{D_{A_\alpha}} = \|\mathbf{u}-\mathbf{u}_n\|^2 + \|\nabla\times(\mathbf{u}-\mathbf{u}_n)\|^2 + \|\nabla\cdot(\mathbf{u}-\mathbf{u}_n)\|^2_{0,\alpha}\\
&= \|\nabla((1-\delta_n(r))\phi)\|^2 + \|\nabla\cdot\nabla((1-\delta_n(r))\phi)\|^2_{0,\alpha}\\
&\leq c\left(\|\delta_n'\phi\|^2_{0,\Omega_n} + \|\nabla\phi\|^2_{0,\Omega_{\tilde{n}}} + \|\Delta\phi\|^2_{0,\alpha,\Omega_{\tilde{n}}} + \|\delta_n''\phi\|^2_{0,\alpha,\Omega_n} + \|r^{-1}\delta_n'\phi\|^2_{0,\alpha,\Omega_n} + \|\delta_n'\nabla\phi\|^2_{0,\alpha,\Omega_n}\right).
\end{aligned}$$

We have the second and third terms in the last line of the above go to 0 by (4.11) and we have $\|r^{-1}\delta_n'\phi\|_{0,\alpha,\Omega_n} \leq c\|\delta''\phi\|_{0,\alpha,\Omega_n}$ by the property of δ_n. Since $|\delta_n'(r)| \leq cn$, $1/(2n) \leq r \leq 1/n$ on Ω_n, and $\alpha \geq 1$, the sixth term in the above is

$$\|\delta_n'\nabla\phi\|^2_{0,\alpha,\Omega_n} \leq c\|r^\alpha n\nabla\phi\|^2_{0,\Omega_n} \leq c\|r^{\alpha-1}\nabla\phi\|^2_{0,\Omega_n} = c\|\nabla\phi\|^2_{0,\Omega_n} \to 0.$$

We now focus on the following two terms: By Lemma 4.3 and $\alpha \geq 1$, for $\epsilon > 0$,

$$\begin{aligned}
&\|\delta_n'\phi\|^2_{0,\Omega_n} + \|\delta_n''\phi\|^2_{0,\alpha,\Omega_n} \leq c\left(\|n\phi\|^2_{0,\Omega_n} + \|r^\alpha n^2\phi\|^2_{0,\Omega_n}\right) \leq c\left(\|r^{-1}\phi\|^2_{0,\Omega_n} + \|r^{\alpha-2}\phi\|^2_{0,\Omega_n}\right)\\
&\leq c(1/2n)^{-2\epsilon}\left(\|r^{-1+\epsilon}\phi\|^2_{0,\Omega_n} + \|r^{\alpha-2+\epsilon}\phi\|^2_{0,\Omega_n}\right) \leq cn^{2\epsilon}\left(\|r^{-1+\epsilon}\phi\|^2_{0,\Omega_{\tilde{n}}} + \|r^{\alpha-2+\epsilon}\phi\|^2_{0,\Omega_{\tilde{n}}}\right)\\
&\leq cn^{2\epsilon}\left(\|r^\epsilon\phi\|^2_{0,\Omega_{\tilde{n}}} + \|r^\epsilon\nabla\phi\|^2_{0,\Omega_{\tilde{n}}} + \|r^{\alpha-1+\epsilon}\phi\|^2_{0,\Omega_{\tilde{n}}} + \|r^{\alpha-1+\epsilon}\nabla\phi\|^2_{0,\Omega_{\tilde{n}}}\right)\\
&\leq cn^{2\epsilon}\left(n^{-2\epsilon}(\|\phi\|^2_{0,\Omega_{\tilde{n}}} + \|\nabla\phi\|^2_{0,\Omega_{\tilde{n}}}) + n^{-2(\alpha-1+\epsilon)}(\|\phi\|^2_{0,\Omega_{\tilde{n}}} + \|\nabla\phi\|^2_{0,\Omega_{\tilde{n}}})\right) \leq c\|\phi\|^2_{1,\Omega_{\tilde{n}}}.
\end{aligned}$$

Hence, we proved that $\|\mathbf{u}-\mathbf{u}_n\|^2_{D_{A_\alpha}} \to 0$ as long as $\alpha > 1-\lambda$. The density $D_{B_\alpha} \cap H^1(\Omega)^3$ in D_{B_α} follows the same process. $\square$

4.2. The existence of H^1-sequences. So far, we have obtained H^1-sequences satisfying (4.6). For given $(\mathbf{E}, s, \mathbf{H}, k)$, we consider the minimization of the functional (3.6) in the partially weighted norm from (2.2),

$$\mathcal{F}^*_\alpha(\mathbf{U}^*; (\mathbf{E}, s, \mathbf{H}, k)) = ||\mathcal{L}^*\mathbf{U}^* - (\mathbf{E}, s, \mathbf{H}, k)||^2_\alpha \tag{4.12}$$

for all $(\mathcal{U}, p, \mathcal{V}, q) \in D(\mathcal{L}^*)$, where the weighted norms involve only the equations corresponding to slack variables s and k. Since s and k are slack variables of the original system, we may assume that $s = 0$ and $k = 0$. Then the corresponding weak form is as follows: Find $\mathbf{U}^* \in D(\mathcal{L}^*)$ satisfying

$$\langle\mathcal{L}^*\mathbf{U}^*, \mathcal{L}^*\mathbf{V}^*\rangle_\alpha = \langle(\mathbf{E}, 0, \mathbf{H}, 0), \mathcal{L}^*\mathbf{V}^*\rangle_\alpha = \langle\mathcal{L}(\mathbf{E}, 0, \mathbf{H}, 0), \mathbf{V}^*\rangle = \langle\mathbf{F}, \mathbf{V}^*\rangle$$

for all $\mathbf{V}^* \in D(\mathcal{L}^*)$, where $\langle\cdot,\cdot\rangle_\alpha = \langle J_\alpha\cdot, J_\alpha\cdot\rangle$ with J_α the diagonal matrix $J_\alpha = \mathrm{diag}[\ 1,\ 1,\ 1,\ r^\alpha,\ 1,\ 1,\ 1,\ r^\alpha\ ]$. As an important step in achieving the goal of this paper, we show that there exists an H^1-sequence, $\{\mathbf{U}_n\}$, satisfying the following.

THEOREM 4.15. *Assume $\alpha > 1-\lambda$. For given $\mathbf{U} = (\mathbf{E}, s, \mathbf{H}, k) \in L^2(\Omega)^8$, there exists a sequence $\mathbf{U}_n \in D(\mathcal{L}^*) \cap H^1(\Omega)^8$ such that*

$$||\mathcal{L}^*\mathbf{U}_n - \mathbf{U}||_\alpha \longrightarrow 0 \tag{4.13}$$

as $n \longrightarrow \infty$.

Proof. By surjectivity of $\mathcal{L}^*$, there exists $\mathbf{U}^* = (\mathcal{U}, \tilde{p}, \mathcal{V}, \tilde{q}) \in D(\mathcal{L}^*)$ such that $\mathcal{L}^*\mathbf{U}^* = \mathbf{U}$. From Theorems 4.12 and 4.13, we have $\mathbf{U}_n \in D(\mathcal{L}^*) \cap H^1(\Omega)^8$ satisfying

(4.14)

$$\nabla \times \mathcal{V}_n - \sigma\nabla q = \mathbf{E} + \sigma\mathcal{U} = \nabla \times \mathcal{V} - \sigma\nabla\tilde{q}\ , \quad ||\nabla \cdot \mathcal{V}_n - a_1\tilde{p} - s||_{0,\alpha} \longrightarrow 0,$$

$$\nabla \times \mathcal{U}_n - \mu\nabla p = \mathbf{H} - \mu\mathcal{V} = \nabla \times \mathcal{U} - \mu\nabla\tilde{p}\ , \quad ||\nabla \cdot \mathcal{U}_n + a_2\tilde{q} - k||_{0,\alpha} \longrightarrow 0,$$

as n goes to infinity, where $\mathbf{U}_n = (\mathcal{U}_n, p, \mathcal{V}_n, q)$ and a_1, a_2 are nonnegative constants. First, consider the case $1-\lambda < \alpha < 1$. By substituting $\mathcal{L}^*\mathbf{U}^* = \mathbf{U}$ into (4.14) and using Lemmas 4.2 and 4.5, we have the first inequality in the following equation:

$$\begin{aligned} ||\mathcal{L}^*\mathbf{U}_n - \mathbf{U}||_\alpha^2 \le c\,\big(\ &||\nabla \times (\mathcal{U}_n - \mathcal{U})||^2 + ||\nabla \cdot (\mathcal{U}_n - \mathcal{U})||_{0,\alpha}^2 \\ &+ ||\nabla \times (\mathcal{V}_n - \mathcal{V})||^2 + ||\nabla \cdot (\mathcal{V}_n - \mathcal{V})||_{0,\alpha}^2\ \big) \\ \le c\ (&||\nabla \cdot (\mathcal{U}_n - \mathcal{U})||_{0,\alpha}^2 + ||\nabla \cdot (\mathcal{V}_n - \mathcal{V})||_{0,\alpha}^2), \end{aligned}$$

where $c = c(\Omega, \mu, \sigma, \alpha)$. Boundary conditions and orthogonality properties provide the second inequality in the above. By (4.14), the right-hand side converges to 0.

Now consider $\alpha \ge 1$. Since $|r| < 1$, it is easy to see that, when $\alpha_1 \ge \alpha_2$, $||\cdot||_{0,\alpha_1} \le ||\cdot||_{0,\alpha_2}$. Therefore, for $\alpha \ge 1$,

$$||\mathcal{L}^*\mathbf{U}_n - \mathbf{U}||_\alpha \ \le\ ||\mathcal{L}^*\mathbf{U}_n - \mathbf{U}||_{1-\epsilon}$$

for $\epsilon > 0$. Hence, the result holds. ∎

COROLLARY 4.16. *Let $\mathbf{U}^* = (\mathcal{U}, \tilde{p}, \mathcal{V}, \tilde{q}) \in D(\mathcal{L}^*)$ satisfying $\mathcal{L}^*\mathbf{U}^* = \mathbf{U}$ and let $\mathbf{U}_n = (\mathcal{U}_n, p, \mathcal{V}_n, q) \in D(\mathcal{L}^*) \cap H^1(\Omega)^8$ satisfying* (4.14)*, where $\mathcal{U}_n = \mathbf{u} + \nabla\delta_n\phi$ and $\mathcal{V}_n = \mathbf{v} + \nabla\delta_n\psi$ with δ_n defined as in* (4.8)*, $\mathbf{u} \in H^1(\Omega)^3 \cap D_A$, $\mathbf{v} \in H^1(\Omega)^3 \cap D_B$, $\phi \in H^1(\Omega)/\mathbb{R}$, and $\psi \in H_0^1(\Omega)$ from Theorems* 4.10 *and* 4.11*. Then*

$$\mathcal{U} = \mathbf{u} + \nabla\phi, \quad \mathcal{V} = \mathbf{v} + \nabla\psi, \quad \tilde{p} = p, \quad \textit{and} \quad \tilde{q} = q.$$

Proof. By taking divergence on the first and third equations in (4.14), we obtain $\tilde{p} = p$ and $\tilde{q} = q$. Then, we have

$$\mathbf{0} = \nabla \times (\mathcal{U} - \mathcal{U}_n) = \nabla \times (\mathcal{U} - (\mathbf{u} + \nabla\delta_n\phi)) = \nabla \times (\mathcal{U} - (\mathbf{u} + \nabla\phi)),$$

$$0 = \nabla \cdot \mathcal{U} + a_2 q - k = \nabla \cdot \mathcal{U} - (\nabla \cdot \mathbf{u} + \Delta\phi) = \nabla \cdot (\mathcal{U} - (\mathbf{u} + \nabla\phi)),$$

which imply $\mathcal{U} = \mathbf{u} + \nabla\phi$. Similarly, $\mathcal{V} = \mathbf{v} + \nabla\psi$. ∎

The singularity on the boundary implies that the solution, $\mathbf{U}^* \in D(\mathcal{L}^*)$, of $\mathcal{L}^*\mathbf{U}^* = \mathbf{U}$ is not in H^1. However, we have shown that there is an H^1-sequence, $\mathbf{U}_n$, satisfying (4.13). This allows us to use the standard H^1-conforming finite elements, as we demonstrate in section 7. In the next theorem we establish the coercivity and continuity of $\mathcal{F}^*$ in the partially weighted norm.

THEOREM 4.17. *If $(\mathcal{U}, p, \mathcal{V}, q) \in D(\mathcal{L}^*)$, then there exist c and C such that*

$$\begin{aligned} &c(||\mathcal{U}|| + ||\nabla \times \mathcal{U}|| + ||r^\alpha \nabla \cdot \mathcal{U}|| + ||p||_1 + ||\mathcal{V}|| + ||\nabla \times \mathcal{V}|| + ||r^\alpha \nabla \cdot \mathcal{V}|| + ||q||_1) \\ \le\ & ||\mathcal{L}^*(\mathcal{U}, p, \mathcal{V}, q)||_\alpha \\ \le\ & C(||\mathcal{U}|| + ||\nabla \times \mathcal{U}|| + ||r^\alpha \nabla \cdot \mathcal{U}|| + ||p||_1 + ||\mathcal{V}|| + ||\nabla \times \mathcal{V}|| + ||r^\alpha \nabla \cdot \mathcal{V}|| + ||q||_1), \end{aligned}$$

where $1 - \lambda < \alpha < 1$.

Proof. It is clear that $||r^\alpha \nabla \cdot \mathcal{U}|| + ||r^\alpha \nabla \cdot \mathcal{V}|| \le ||\mathcal{L}^*(\mathcal{U}, p, \mathcal{V}, q)||_\alpha$. By Lemmas 4.2 and 4.5, and the Poincaré inequality, it is enough to show that

$$||\nabla \times \mathcal{U}|| + ||\nabla p|| + ||\nabla \times \mathcal{V}|| + ||\nabla q|| \le c||\mathcal{L}^*(\mathcal{U}, p, \mathcal{V}, q)||_\alpha.$$

Using orthogonality and Hölder's inequality, we can easily show the lower inequality. The upper inequality follows by the triangle inequality. For more details, see [17]. ◻

5. Scaling in FOSLL*. In this section, we briefly introduce scaling in FOSLS and FOSLL*. From [18], it is known that using a scaling in FOSLS and FOSLL* sometimes has computational advantages. Here, we are particularly interested in scaling with $\sqrt{\mu}$ and $\sqrt{\sigma}$ since it gives orthogonality between $\nabla\times$ and ∇ in FOSLL*.

The eddy current equations (3.1) can be rewritten as

$$\mathcal{L}_s \mathbf{U} = \begin{bmatrix} -\sqrt{\sigma} I & 0 & \nabla \times \frac{1}{\sqrt{\mu}} & -\nabla \frac{1}{\sqrt{\mu}} \\ 0 & -\frac{1}{\sqrt{\sigma}} a_1 & \nabla \cdot \sqrt{\mu} & 0 \\ \nabla \times \frac{1}{\sqrt{\sigma}} & -\nabla \frac{1}{\sqrt{\sigma}} & \sqrt{\mu} I & 0 \\ \nabla \cdot \sqrt{\sigma} & 0 & 0 & \frac{1}{\sqrt{\mu}} a_2 \end{bmatrix} \begin{bmatrix} \sqrt{\sigma}\mathbf{E} \\ \sqrt{\sigma} s \\ \sqrt{\mu}\mathbf{H} \\ \sqrt{\mu} k \end{bmatrix} = \mathbf{F}. \tag{5.1}$$

Then, the corresponding dual problem has the form

$$\mathcal{L}_s^* \mathbf{U}^* = \begin{bmatrix} -\sqrt{\sigma} I & 0 & \frac{1}{\sqrt{\sigma}} \nabla\times & -\sqrt{\sigma}\nabla \\ 0 & -\frac{1}{\sqrt{\sigma}} a_1 & \frac{1}{\sqrt{\sigma}} \nabla\cdot & 0 \\ \frac{1}{\sqrt{\mu}} \nabla\times & -\sqrt{\mu}\nabla & \sqrt{\mu} I & 0 \\ \frac{1}{\sqrt{\mu}} \nabla\cdot & 0 & 0 & \frac{1}{\sqrt{\mu}} a_2 \end{bmatrix} \begin{bmatrix} \mathcal{U} \\ p \\ \mathcal{V} \\ q \end{bmatrix} = \begin{bmatrix} \sqrt{\sigma}\mathbf{E} \\ \sqrt{\sigma} s \\ \sqrt{\mu}\mathbf{H} \\ \sqrt{\mu} k \end{bmatrix}. \tag{5.2}$$

FOSLL* for the scaled system minimizes the dual functional $\mathcal{F}_s^*(\mathbf{U}^*; \mathbf{U}) = ||\mathcal{L}_s^* \mathbf{U}^* - \mathbf{U}||^2$ in the weak sense as follows: Find $\mathbf{U}^* \in D(\mathcal{L}^*)$ that satisfies

$$\langle \mathcal{L}_s^* \mathbf{U}^*, \mathcal{L}_s^* \mathbf{V}^* \rangle = \langle \mathbf{U}, \mathcal{L}_s^* \mathbf{V}^* \rangle = \langle \mathcal{L}_s \mathbf{U}, \mathbf{V}^* \rangle = \langle \mathbf{F}, \mathbf{V}^* \rangle \tag{5.3}$$

for all $\mathbf{V}^* \in D(\mathcal{L}^*)$. To gain insight into the effectiveness of the scaled approach in FOSLL*, we observe the formal normal, $\mathcal{L}_s \mathcal{L}_s^*$, of (5.3):

$$\begin{bmatrix} \sigma I + \nabla\times\frac{1}{\mu}\nabla\times - \nabla\frac{1}{\mu}\nabla\cdot & 0 & 0 & \sigma\nabla - \nabla\frac{a_2}{\mu} \\ 0 & \frac{a_1^2}{\sigma} - \nabla\cdot\mu\nabla & \nabla\cdot\mu - \frac{a_1}{\sigma}\nabla\cdot & 0 \\ 0 & \nabla\frac{a_1}{\sigma} - \mu\nabla & \nabla\times\frac{1}{\sigma}\nabla\times - \nabla\frac{1}{\sigma}\nabla\cdot + \mu I & 0 \\ \frac{a_2}{\mu}\nabla\cdot - \nabla\cdot\sigma & 0 & 0 & \frac{a_2^2}{\mu} - \nabla\cdot\sigma\nabla \end{bmatrix}.$$

Compare the above to the formal normal, $\mathcal{L}\mathcal{L}^*$, of the original system (3.1). The formal normal of the scaled system provides two small systems, each totally separated, corresponding to the variables $(\mathcal{U}, q)$ and $(\mathcal{V}, p)$, respectively:

$$\begin{bmatrix} \sigma I + \nabla \times \frac{1}{\mu} \nabla \times - \nabla \frac{1}{\mu} \nabla \cdot & \sigma \nabla - \nabla \frac{a_2}{\mu} \\ \frac{a_2}{\mu} \nabla \cdot - \nabla \cdot \sigma & \frac{a_2^2}{\mu} - \nabla \cdot \sigma \nabla \end{bmatrix} \begin{bmatrix} \mathcal{U} \\ q \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix} \tag{5.4}$$

and

$$\begin{bmatrix} \mu I + \nabla \times \frac{1}{\sigma} \nabla \times - \nabla \frac{1}{\sigma} \nabla \cdot & \nabla \frac{a_1}{\sigma} - \mu \nabla \\ \nabla \cdot \mu - \frac{a_1}{\sigma} \nabla \cdot & \frac{a_1^2}{\sigma} - \nabla \cdot \mu \nabla \end{bmatrix} \begin{bmatrix} \mathcal{V} \\ p \end{bmatrix} = \begin{bmatrix} \mu \mathbf{H}_{\text{old}} \\ 0 \end{bmatrix}. \tag{5.5}$$

The weak form also separates and we solve two smaller systems. For the eddy current problem, it is clear that $(\mathcal{U}, q) = (\mathbf{0}, 0)$. For more general formulations, both systems might have a nontrivial solution.

Remark 5.1. If σ, μ are constants and $a_1 = a_2 = \sigma \cdot \mu$, then (5.5) is reduced to

$$\begin{bmatrix} \mu I + \frac{1}{\sigma} (\nabla \times \nabla \times - \nabla \nabla \cdot) & 0 \\ 0 & \sigma \mu^2 - \mu \nabla \cdot \nabla \end{bmatrix} \begin{bmatrix} \mathcal{V} \\ p \end{bmatrix} = \begin{bmatrix} \mu \mathbf{H}_{\text{old}} \\ 0 \end{bmatrix}.$$

Clearly, $p = 0$ and $\mathcal{V}$ satisfies

$$\mu \mathcal{V} + \frac{1}{\sigma} \nabla \times \nabla \times \mathcal{V} - \frac{1}{\sigma} \nabla \nabla \cdot \mathcal{V} \ = \ \mu \mathbf{H}_{\text{old}}.$$

The above equation is the same as a modified Galerkin formulation for the magnetic field, $\mathbf{H}$. In the context of constant σ and μ, using FOSLL* with the square root scaling described in (5.1) and certain values for a_1, a_2 is equivalent to solving the original problem (2.6) by eliminating the electric field, $\mathbf{E}$, and using a modified Galerkin formulation on $\mathbf{H}$. However, it is the case of nonconstant σ and μ and the presence of reentrant edges that we consider in this paper.

Remark 5.2. In the modified FOSLL*, the formal normal of (5.3) is

$$\begin{bmatrix} \sigma I + \nabla \times \frac{1}{\mu} \nabla \times - \nabla \frac{r^{2\alpha}}{\mu} \nabla \cdot & 0 & 0 & \sigma \nabla - \nabla \frac{r^{2\alpha} a_2}{\mu} \\ 0 & \frac{r^{2\alpha} a_1^2}{\sigma} - \nabla \cdot \mu \nabla & \nabla \cdot \mu - \frac{r^{2\alpha} a_1}{\sigma} \nabla \cdot & 0 \\ 0 & \nabla \frac{r^{2\alpha} a_1}{\sigma} - \mu \nabla & \mu I + \nabla \times \frac{1}{\sigma} \nabla \times - \nabla \frac{r^{2\alpha}}{\sigma} \nabla \cdot & 0 \\ \frac{r^{2\alpha} a_2}{\mu} \nabla \cdot - \nabla \cdot \sigma & 0 & 0 & \frac{r^{2\alpha} a_2^2}{\mu} - \nabla \cdot \sigma \nabla \end{bmatrix}.$$

Because of the weighting terms, there is no simple way to further decouple the equations through a choice of a_1 and a_2. The term in the (3,3) position in the above is similar to the formal normal associated with the partially weighted modified Galerkin described in [10].

6. Discrete approximation. Let $\mathcal{T}_h$ be a partition of the domain $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$, and each finite element $K \in \mathcal{T}_h$ be a closed subset of $\overline{\Omega}$ with $h := \max\{h_K := diam(K) : K \in \mathcal{T}_h\}$. Assume that the partition $\mathcal{T}_h$ is regular so that we can choose a finite element basis that is conforming and satisfies the approximation property (see [6]). We also assume that there exists a constant, ρ, satisfying $h \leq \rho h_K$. Define by P_k the space of all polynomials of degree $\leq k$ with respect to each variable. Let the standard polynomial interpolation operator, $I^h \in \mathcal{L}((H^1(\Omega))^8; (H^1(\Omega))^8)$, be such that $I^h p = p$ for all $p \in (P_1)^8$, and let the finite dimensional subspace, $\mathcal{W}^h \subset D(\mathcal{L}^*) \cap H^1(\Omega)^8$, have $I^h(D(\mathcal{L}^*) \cap H^1(\Omega)^8 \cap \mathcal{C}^0(\Omega)) \subset \mathcal{W}^h$.

From section 3, we know that, for given $\mathbf{U} \in L^2(\Omega)^8$, there exists the solution $\mathbf{U}^* \in D(\mathcal{L}^*)$ satisfying $\mathcal{L}^*\mathbf{U}^* = \mathbf{U}$, that is,

$$\mathbf{U}^* = \arg \min_{\mathcal{X} \in D(\mathcal{L}^*)} ||\mathcal{L}^*\mathcal{X} - \mathbf{U}||_\alpha. \tag{6.1}$$

Here, we minimize in (6.1) over a finite-dimensional subspace $\mathcal{W}^h$ which yields the corresponding weak form as follows: Find $\mathbf{U}^h \in \mathcal{W}^h$ satisfying

$$\left\langle \mathcal{L}^*\mathbf{U}^h, \mathcal{L}^*\mathbf{X}^h \right\rangle_\alpha = \left\langle \mathbf{U}, \mathcal{L}^*\mathbf{X}^h \right\rangle_\alpha = \left\langle (\mathbf{0}, 0, \mu\mathbf{H}_{\text{old}}, 0), \mathbf{X}^h \right\rangle \tag{6.2}$$

for all $\mathbf{X}^h \in \mathcal{W}^h$. By computing $\mathcal{L}^*\mathbf{U}^h$, we obtain the approximations for $\mathbf{E}$ and $\mathbf{H}$:

$$\mathbf{E}^h = -\sigma\mathcal{U}^h + \nabla \times \mathcal{V}^h - \sigma\nabla\tilde{q}^h, \qquad \mathbf{H}^h = \nabla \times \mathcal{U}^h - \mu\nabla\tilde{p}^h + \mu\mathcal{V}^h, \tag{6.3}$$

where $\mathbf{U}^h = \left(\mathcal{U}^h, \tilde{p}^h, \mathcal{V}^h, \tilde{q}^h\right)$.

The following theorem provides the L^2-error estimates for the solution $\mathbf{E}$ and $\mathbf{H}$ of (2.6) with the approximation $\mathcal{L}^*\mathbf{U}^h$. Here, we use Theorem 4.15 to accomplish the L^2-error estimates by adopting the standard finite element approximation property. Vectors $(\mathbf{E}, s, \mathbf{H}, k)$, $(\mathcal{U}, p, \mathcal{V}, q), (\mathcal{U}_n, p, \mathcal{V}_n, q)$, and $(\mathcal{U}_n^h, p^h, \mathcal{V}_n^h, q^h)$ are abbreviated to $\mathbf{U}$, $\mathbf{U}^*, \mathbf{U}_n$, and $\mathbf{U}_n^h$, respectively.

THEOREM 6.1. *Assume* $\mathbf{U} \in D(\mathcal{L})$ *and* $\alpha > 1 - \lambda$. *Let* $\mathbf{U}^* = (\mathcal{U}, p, \mathcal{V}, q) \in D(\mathcal{L}^*)$ *such that* $\mathcal{L}^*\mathbf{U}^* = \mathbf{U}$. *Then, Corollary* 4.16 *leads the decompositions* $\mathcal{U} = \mathbf{u} + \nabla\phi$ *and* $\mathcal{V} = \mathbf{v} + \nabla\psi$. *Assume* $\mathbf{u}, \mathbf{v} \in H^{1+\eta_1}(\Omega)^3$ *and* $p, q \in H^{1+\eta_2}(\Omega)$ *for some* $\eta_1, \eta_2 > 0$. *If* $\mathbf{U}^h \in \mathcal{W}^h$ *satisfies* (6.2), *then there exists a constant* c *such that*

$$||\mathbf{U} - \mathcal{L}^*\mathbf{U}^h||_\alpha^2 \le c\ h^{2\tau}\big(|\mathbf{u}|_{1+\eta_1}^2 + |\mathbf{v}|_{1+\eta_1}^2 + ||\phi||_{3,1+\beta}^2 + ||\psi||_{3,1+\beta}^2 + |p|_{1+\eta_2}^2 + |q|_{1+\eta_2}^2\big)$$

for any $\tau < \min\left\{\eta_1, \eta_2, \frac{\alpha-1+\lambda}{\alpha+1}\right\}$ *and some* $\beta \in (1-\lambda, 1)$, $\beta < \alpha$.

Proof. Let $\mathbf{U}_n \in D(\mathcal{L}^*) \cap H^1(\Omega)^8$ satisfying Theorem 4.15, and let

$$\mathbf{U}_n^h = (\mathcal{U}_n^h, p^h, \mathcal{V}_n^h, q^h) = \arg \min_{\mathcal{X}_n^h \in \mathcal{W}^h} ||\mathcal{L}^*\mathbf{U}_n - \mathcal{L}^*\mathcal{X}_n^h||_\alpha. \tag{6.4}$$

By the triangle inequality,

$$||\mathbf{U} - \mathcal{L}^*\mathbf{U}^h||_\alpha^2 \le 3\left(||\mathcal{L}^*\mathbf{U}^* - \mathcal{L}^*\mathbf{U}_n||_\alpha^2 + ||\mathcal{L}^*\mathbf{U}_n - \mathcal{L}^*\mathbf{U}_n^h||_\alpha^2 + ||\mathcal{L}^*\mathbf{U}_n^h - \mathcal{L}^*\mathbf{U}^h||_\alpha^2\right).$$

From Theorems 4.12, 4.13, and 4.15, we have

$$||\mathbf{U} - \mathcal{L}^*\mathbf{U}_n||_\alpha^2 \ < \ c\ n^{-2(\alpha-\beta)}\left(||\phi||_{2,\beta}^2 + ||\psi||_{2,\beta}^2\right). \tag{6.5}$$

The linearity of $\mathcal{L}^*$ and the optimality on the finite-dimensional space imply

$$\begin{aligned} ||\mathcal{L}^*\mathbf{U}_n^h - \mathcal{L}^*\mathbf{U}^h||_\alpha^2 &= \left\langle \mathcal{L}^*(\mathbf{U}_n^h - \mathbf{U}_n + \mathbf{U}_n - \mathbf{U}^* + \mathbf{U}^* - \mathbf{U}^h), \mathcal{L}^*(\mathbf{U}_n^h - \mathbf{U}^h)\right\rangle_\alpha \\ &\le ||\mathcal{L}^*\mathbf{U}_n - \mathcal{L}^*\mathbf{U}^*||_\alpha\ ||\mathcal{L}^*\mathbf{U}_n^h - \mathcal{L}^*\mathbf{U}^h||_\alpha. \end{aligned} \tag{6.6}$$

Thus, (6.5) and (6.6) yield

$$||\mathbf{U} - \mathcal{L}^*\mathbf{U}^h||_\alpha^2 \le c\ n^{-2(\alpha-\beta)}\left(||\phi||_{2,\beta}^2 + ||\psi||_{2,\beta}^2\right) + c\ ||\mathcal{L}^*\mathbf{U}_n - \mathcal{L}^*\mathbf{U}_n^h||_\alpha^2. \tag{6.7}$$

Since $\mathbf{U}_n^h$ satisfies (6.4), by Céa's lemma, $||\mathcal{L}^*\mathbf{U}_n - \mathcal{L}^*\mathbf{U}_n^h||_\alpha^2 \le c\, ||\mathcal{L}^*\mathbf{U}_n - \mathcal{L}^* I^h \mathbf{U}_n||_\alpha^2$. Using the triangle inequality, we have

$$\begin{aligned} ||\mathcal{L}^*\mathbf{U}_n - \mathcal{L}^* I^h\mathbf{U}_n||_\alpha^2 &\le c\left(||\mathcal{U}_n - I^h\mathcal{U}_n||^2 + ||\mathcal{V}_n - I^h\mathcal{V}_n||^2\right.\\ &\quad + ||\nabla\times(\mathcal{U}_n - I^h\mathcal{U}_n)||^2 + ||\nabla\cdot(\mathcal{U}_n - I^h\mathcal{U}_n)||_{0,\alpha}^2 + ||\nabla(p - I^h p)||^2\\ &\quad \left.+||\nabla\times(\mathcal{V}_n - I^h\mathcal{V}_n)||^2 + ||\nabla\cdot(\mathcal{V}_n - I^h\mathcal{V}_n)||_{0,\alpha}^2 + ||\nabla(q - I^h q)||^2\right). \end{aligned} \tag{6.8}$$

First, we consider $\mathcal{U}_n$-terms. By [12], we have

$$\begin{aligned} ||\mathcal{U}_n - I^h\mathcal{U}_n||^2 + ||\nabla\times(\mathcal{U}_n - I^h\mathcal{U}_n)||^2 + ||\nabla\cdot(\mathcal{U}_n - I^h\mathcal{U}_n)||_{0,\alpha}^2 \\ \le c||\nabla(\mathcal{U}_n - I^h\mathcal{U}_n)||^2 = c\sum_{K\in\mathcal{T}_h} ||\nabla(\mathcal{U}_n - I^h\mathcal{U}_n)||_K^2, \end{aligned}$$

where $||\cdot||_K$ means an integration over K. Since ϕ satisfies

$$\left\{\begin{array}{rcll} \nabla\cdot\nabla\phi &=& -\nabla\cdot\mathbf{u} - a_2\tilde{q} + k & \text{in } \ \Omega,\\ \phi &=& 0 & \text{on } \ \partial\Omega, \end{array}\right. \tag{6.9}$$

and $\nabla\cdot\mathbf{u} + a_2\tilde{q} - k \in H_\beta^1(\Omega) \subset H_{1+\beta}^1(\Omega)$, the solution, ϕ, of (6.9) is in $H_{1+\beta}^3(\Omega)$ (see [19]). From Theorems 4.12 and 4.15, $\mathcal{U}_n$ is decomposed of $\mathbf{u}+\nabla\delta_n\phi$, where δ_n is defined as in (4.8). The fact that $\phi \in H_{1+\beta}^3(\Omega)$ and the definition of δ_n yield $\delta_n\phi \in H^3(\Omega)$. On each element K, we use the triangle inequality and standard interpolation error estimates to obtain

$$\begin{aligned} ||\nabla(\mathcal{U}_n - I^h\mathcal{U}_n)||_K^2 &\le c\left(||\nabla(\mathbf{u} - I^h\mathbf{u})||_K^2 + ||\nabla(\nabla\phi_n - I^h\nabla\phi_n)||_K^2\right)\\ &\le c\ h^{2\eta_1}|\mathbf{u}|_{1+\eta_1,K}^2 + c\ h^2|\phi_n|_{3,K}^2. \end{aligned}$$

Since $\delta_n = 0$ when $r \le (1/2n)$ and $\delta_n' = 0$ when $r \notin (1/2n, 1/n)$,

$$\begin{aligned} &\sum_K |\phi_n|_{3,K}^2 = |\phi_n|_3^2 \le c\int |\delta_n'''\phi|^2 + |\delta_n''\nabla\phi|^2 + |\delta_n'\nabla^2\phi|^2 + |\delta_n\nabla^3\phi|^2 d\Omega\\ &\le c\iiint_{\frac{1}{2n}}^{\frac{1}{n}} |n^3\phi|^2 + |n^2\nabla\phi|^2 + |n\nabla^2\phi|^2 r\,dr\,d\theta\,dz + c\iiint_{\frac{1}{2n}}^{R(\theta)} |\nabla^3\phi|^2 r\,dr\,d\theta\,dz\\ &\le cn^{2(1+\beta)}\left(\iiint_{\frac{1}{2n}}^{\frac{1}{n}}\sum_{k=0}^{2} |r^{\beta-k}\nabla^{2-k}\phi|^2 d\Omega + \iiint_{\frac{1}{2n}}^{R(\theta)} |r^{1+\beta}\nabla^3\phi|^2 d\Omega\right) \le cn^{2(1+\beta)}||\phi||_{3,1+\beta}^2. \end{aligned}$$

Thus, we have $||\nabla(\mathcal{U}_n - I^h\mathcal{U}_n)||^2 \le\ c\big(h^{2\eta_1}|\mathbf{u}|_{1+\eta_1}^2 + h^2 n^{2(1+\beta)}||\phi||_{3,1+\beta}^2\big)$. Choose n such that $\frac{1}{2n} < \sqrt{2}h^{\frac{1}{\alpha+1}} < \frac{11}{20n}$ to balance with (6.7). Then, the optimal choice of β is $1-\lambda+\epsilon$ and this yields $hn^{1+\beta} = h^{\frac{\alpha-1+\lambda}{\alpha+1}-\epsilon}$. Then,

$$||\nabla(\mathcal{U}_n - I^h\mathcal{U}_n)||^2 \le\ c\left(h^{2\eta_1}|\mathbf{u}|_{1+\eta_1}^2 + h^{2\frac{\alpha-1+\lambda}{\alpha+1}-\epsilon}||\phi||_{3,1+\beta}^2\right).$$

The above calculation can be applied to $\mathcal{V}_n$ analogously. For p and q, the standard error estimates yields $||\nabla(p - I^h p)||^2 + ||\nabla(q - I^h q)||^2 \le ch^{2\eta_2}(|p|_{1+\eta_2}^2 + |q|_{1+\eta_2}^2)$. $\square$

COROLLARY 6.2. *If μ, σ are constants, then η_1, η_2 are any real values $< \lambda$.*

Proof. If μ and σ are constants, then $\nabla\times\nabla\times(\mathbf{u}+\nabla\phi)\in L^2(\Omega)$, where $\mathbf{u}$ and ϕ are from the proof of Theorem 6.1. Also, $\nabla\cdot(\mathbf{u}+\nabla\phi)=0$ and $\mathbf{n}\times(\mathbf{u}+\nabla\phi)=0$ on $\partial\Omega$. Thus, by [9], we have $\mathbf{u}\in H^{1+\eta_1}(\Omega)^3$ for any $\eta_1<\lambda$. The variable p is the solution of the Poisson equation with a Dirichlet boundary condition. Thus, $p\in H^{1+\eta_2}(\Omega)$, where $\eta_2<\lambda$. Similarly, we have $\mathbf{v}\in H^{1+\eta_1}(\Omega)^3$ and $q\in H^{1+\eta_2}(\Omega)$ for any $\eta_1,\eta_2<\lambda$. ◻

Remark 6.3. In [10], error estimates in the D_{A_α}-norm (see (4.9)) with higher regularity in $\mathbf{E}$ were developed. They used H^1-conforming finite element spaces which include $\nabla\Phi^h$, where Φ^h is an almost affine family of $\mathcal{C}^1$ elements and has good approximation properties in the H^2_β-norm. In this paper, we use H^1-conforming finite elements to approximately solve the problem and develop L^2-error estimates. Our approximation to the electric field is of the form $E^h=-\sigma\mathcal{U}^h+\nabla\times\mathcal{V}^h-\sigma\nabla\tilde{q}^h$, where $\mathcal{U}^h,\mathcal{V}^h$, and $\tilde{q}^h$ are chosen from H^1-conforming finite element spaces, which means we explicitly present the solution as a combination of such terms, and thus, do not need to construct special finite element spaces.

In the following section, we present several numerical examples. The results show clearly that the convergence rate is related to α values as well as to the regularity of the dual solution in agreement with the above theorem.

7. Numerical results. In this section, we report on numerical results of applying the modified FOSLL* method to problem (3.1). We choose the prototype domain described by

$$\Omega=(-0.5,0.5)^3\backslash\{(x,y,z)|0\le x\le 0.5,-0.5\le y\le 0,-0.5<z<0.5\}.$$

The domain has a reentrant edge along the z-axis with interior angle $\frac{3\pi}{2}$. Thus, we expect the solution to have a singularity of the form $r^{-\frac{1}{3}}$, where r is the distance to the z-axis. The square root scaling described in section 5 was used for all three tests. This requires solving for only four dependent variables, denoted by $(\mathcal{V},p)$, since the other four variables $(\mathcal{U},q)$ are known to be zero. Trilinear finite elements were used for all variables. In this context, we minimize $||\mathcal{L}^*\mathcal{X}^h-(\mathbf{E},0,\mathbf{H},0)||_\alpha$ over $\mathcal{X}^h=(\mathcal{U}^h,p^h,\mathcal{V}^h,q^h)$ in the finite-dimensional subspace $\mathcal{W}^h$, holding $(\mathcal{U}^h,q^h)=(\mathbf{0},0)$, in order to get the approximation, $\mathbf{U}^h$, for the dual solution, $\mathbf{U}^*$, of (3.5). Then, we compute $\mathcal{L}^*\mathbf{U}^h$ as the approximation for $(\mathbf{E},0,\mathbf{H},0)$ and observe the L^2-errors $||\mathbf{E}-\mathbf{E}^h||$ and $||\mathbf{H}-\mathbf{H}^h||$.

The software package FOSPACK [22] was used to construct the discrete systems and to solve them by a conjugate gradient iteration preconditioned by algebraic multigrid (AMG) using W(1,1)-cycles. Problems with given exact solutions were constructed so that the error could be monitored. The constants a_1 and a_2 were fixed at 0. However, the results are similar to those achieved when they are fixed as positive constants. A residual reduction 10^{-10} was used as the AMG W-cycle stopping criterion. While this level of error is excessive in practice, we employ it here to remove algebraic error from the calculation of the convergence of the discrete solution.

Example 7.1. We choose the exact solutions $\mathbf{E}$ and $\mathbf{H}$ to be

$$\mathbf{E}=\frac{1}{\sigma}\nabla\times\mathbf{H}\quad\text{and}\quad\mathbf{H}=(\ \partial_y g,\ \ -\partial_x g,\ \ 0\ ),$$

where

$$g=\delta(r)r^{\frac{2}{3}}\sin\left(\frac{2}{3}\theta\right)\sin(2\pi z)\quad\text{and}\quad\delta(r)=\begin{cases}1, & r\le 0.25,\\ 0, & r\ge 0.375\end{cases}$$

TABLE 7.1
The L^2-norm of the errors and observed convergence rates, τ, for Example 7.1 ($\times 10^{-1}$ means that the values in the table divide by 10), $\|\mathbf{E}\| \sim 10.290$, $\|\mathbf{H}\| \sim 0.55727$.

$\|\|\mathbf{E}-\mathbf{E}^h\|\|$										
	$\alpha=0$		$\alpha=2/3$		$\alpha=4/3$		$\alpha=2$		$\alpha=3$	
1/8	4.67	τ	4.65	τ	4.64	τ	4.64	τ	4.63	τ
1/16	3.91	0.26	3.84	0.28	3.80	0.29	3.79	0.29	3.79	0.29
1/32	2.16	0.85	1.97	0.96	1.91	0.99	1.89	1.00	1.88	1.01
1/64	1.45	0.57	1.08	0.86	1.00	0.94	0.97	0.96	0.96	0.97
$\|\|\mathbf{H}-\mathbf{H}^h\|\|$ ($\times 10^{-1}$)										
	$\alpha=0$		$\alpha=2/3$		$\alpha=4/3$		$\alpha=2$		$\alpha=3$	
1/8	2.32	τ	2.18	τ	2.11	τ	2.06	τ	2.02	τ
1/16	1.74	0.41	1.36	0.68	1.11	0.93	2.06	1.06	2.02	1.11
1/32	1.57	0.16	0.92	0.56	0.55	1.00	0.45	1.14	0.41	1.19
1/64	1.52	0.04	0.69	0.42	0.31	0.85	0.23	0.94	0.21	0.94

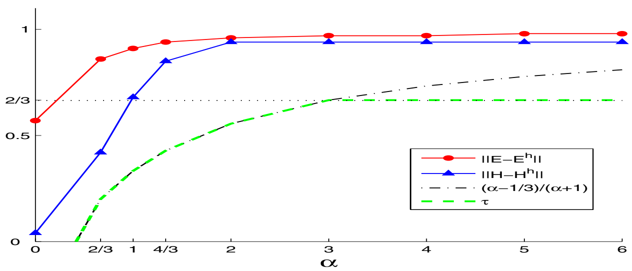

FIG. 7.1. *Finite element convergence rate, τ, as a function of α for Example 7.1.*

TABLE 7.2
AMG convergence factors for Example 7.1.

	$\alpha=0$	$\alpha=2/3$	$\alpha=4/3$	$\alpha=2$	$\alpha=3$	$\alpha=4$	$\alpha=5$	$\alpha=6$
1/8	0.03	0.03	0.04	0.05	0.05	0.06	0.07	0.07
1/16	0.03	0.05	0.09	0.14	0.28	0.23	0.20	0.20
1/32	0.03	0.17	0.20	0.29	0.33	0.37	0.42	0.44
1/64	0.03	0.14	0.32	0.40	0.44	0.51	0.54	0.54

with $r = \sqrt{x^2+y^2}$, $\theta = \arctan(\frac{y}{x})$, and $\delta(r) \in \mathcal{C}^3$ cut-off function. Then, the solution satisfies type II boundary conditions. We fix the $\mu = 1$ and $\sigma = 1$.

Table 7.1 displays the L^2-errors of $\mathbf{E}$ and $\mathbf{H}$. The rate, τ, represents the value of the observed convergent factor, h^τ, when the mesh decreases from h to $h/2$. As shown in Table 7.1, standard FOSLL* ($\alpha = 0$) gives poor convergence. The declines in convergence factors are dramatic in this case. This is to be expected because the exact dual solutions $\mathcal{U}$ and $\mathcal{V}$ are not in H^1, but rather in H^γ for any $\gamma < \frac{2}{3}$. The results in Table 7.1 for $\alpha > 1-\lambda = \frac{1}{3}$ show that partial unweighting of the functional produces improved convergence in all terms of the functional. By Theorem 6.1, the L^2-errors of $\mathbf{E}$ and $\mathbf{H}$ are expected to exhibit $O(h^\tau)$, for any $\tau < \min\{\frac{2}{3}, \frac{\alpha-\frac{1}{3}}{\alpha+1}\}$ (dashed line in Figure 7.1) as long as $\alpha > \frac{1}{3}$, that is, the bound τ, on the convergence rate stays at $\frac{2}{3}$ for $\alpha > 3$. In fact, the results show better convergence than expected. In Figure 7.1, we compare convergence rates for the L^2-errors in $\mathbf{E}$ and $\mathbf{H}$ while the mesh moves from 1/32 to 1/64 with more α values than are showed in Table 7.1. We observe in Table 7.2 that increasing α results in an increasing convergence factor for

TABLE 7.3
The L^2-norm of the errors and observed convergence rates, τ, for Example 7.2, $\|\mathbf{E}\| \sim 1.9302$, $\|\mathbf{H}\| \sim 0.55727$.

$\|\|\mathbf{E}-\mathbf{E}^h\|\|$ $(\times 10^{-1})$										
	$\alpha = 0$		$\alpha = 2/3$		$\alpha = 4/3$		$\alpha = 2$		$\alpha = 3$	
1/8	9.56	τ	9.27	τ	9.13	τ	9.04	τ	8.96	τ
1/16	8.57	0.16	7.75	0.26	7.34	0.31	7.20	0.33	7.14	0.33
1/32	6.77	0.34	5.06	0.62	4.49	0.71	4.26	0.76	4.04	0.82
1/64	6.20	0.13	3.77	0.43	3.07	0.55	2.70	0.66	2.38	0.76
$\|\|\mathbf{H}-\mathbf{H}^h\|\|$ $(\times 10^{-1})$										
	$\alpha = 0$		$\alpha = 2/3$		$\alpha = 4/3$		$\alpha = 2$		$\alpha = 3$	
1/8	2.34	τ	2.24	τ	2.18	τ	2.14	τ	2.09	τ
1/16	1.83	0.36	1.60	0.48	1.38	0.65	1.22	0.81	1.09	0.94
1/32	1.71	0.10	1.33	0.26	0.93	0.58	0.68	0.84	0.52	1.05
1/64	1.68	0.02	1.18	0.18	0.64	0.53	0.39	0.81	0.26	0.98

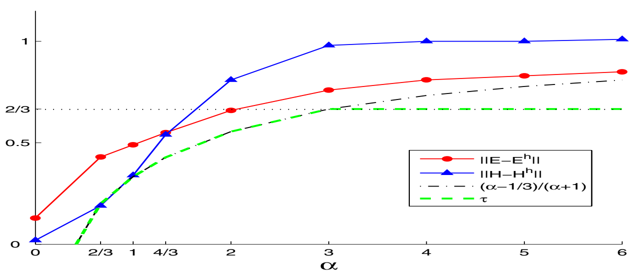

FIG. 7.2. *Finite element convergence rate, τ, as a function of α for Example* 7.2.

the AMG algorithm. This behavior is dependent on the particular AMG algorithm that was used in the test. An improved AMG would change the picture.

Example 7.2. In this example, we take a smooth function for the coefficient σ. Let $\mathbf{E}$ and $\mathbf{H}$ be the same as in Example 7.1 and let $\mu = 0.5$ and $\sigma = 100(x^2+y^2)+1$.

Table 7.3 shows the L^2-errors of $\mathbf{E}$ and $\mathbf{H}$ and the convergence rates. More convergence rates corresponding to α values when the mesh moves from $1/32$ to $1/64$ appear in Figure 7.2. Note that the observed convergence rates are slightly worse than the ones in Example 7.1. The AMG convergence factor behaves essentially the same as in the first example.

In the next example, we examine the case having discontinuous coefficients as well as a reentrant edge on the boundary.

Example 7.3. Let $\mathbf{E}$ and $\mathbf{H}$ be the same as in Example 7.1. Let $\mu = \sigma = 1$ if $r = \sqrt{x^2+y^2} \le 0.25$ and $\mu = 25,\ \sigma = 100$ otherwise.

In this example, we need to be careful about the regularity of $\mathbf{E}$ and $\mathbf{H}$. Since μ and σ have jumps at $r = 0.25$, $\mathbf{E}$ is not in $H(\nabla\times)$ but in $H(\nabla\times\sigma)$, and $\mathbf{H}$ is not in $H(\nabla\cdot\mu)$ but in $H(\nabla\cdot)$. $\mathbf{E}$ and $\mathbf{H}$ do not satisfy the eddy current equations, but are useful as a test to observe how modified FOSLL* would work for a problem with both discontinuous coefficients and a reentrant edge. Numerical results in Table 7.4 show great convergence with modified FOSLL* approximation even though the problem has both nongrid-aligned discontinuities in the coefficients and a boundary singularity. Convergence rates of the L^2-errors for $\mathbf{E}$ and $\mathbf{H}$ are greater than both of $\frac{2}{3}$ and $\frac{\alpha-\frac{1}{3}}{\alpha+1}$ for $\alpha > 3$. Figure 7.3 shows convergence rates for more values of α using grid size $h = 1/64$.

TABLE 7.4
The L^2-norm of the errors and observed convergence rates, τ, for Example 7.3, $\|\mathbf{E}\| \sim 8.1056$, $\|\mathbf{H}\| \sim 0.55727$.

$\|\|\mathbf{E}-\mathbf{E}^h\|\|$										
	$\alpha=0$		$\alpha=2/3$		$\alpha=4/3$		$\alpha=2$		$\alpha=3$	
1/8	2.37	τ	2.35	τ	2.34	τ	2.34	τ	2.34	τ
1/16	2.16	0.14	2.10	0.16	2.08	0.17	2.07	0.18	2.07	0.18
1/32	1.19	0.86	1.06	0.99	1.02	1.02	1.02	1.03	1.01	1.03
1/64	0.82	0.54	0.59	0.83	0.55	0.89	0.54	0.91	0.53	0.92
$\|\|\mathbf{H}-\mathbf{H}^h\|\|$ $(\times 10^{-2})$										
	$\alpha=0$		$\alpha=2/3$		$\alpha=4/3$		$\alpha=2$		$\alpha=3$	
1/8	20.7	τ	20.3	τ	20.1	τ	19.9	τ	19.8	τ
1/16	11.9	0.80	10.4	0.97	9.60	1.07	9.23	1.11	8.99	1.14
1/32	9.35	0.35	6.22	0.74	4.89	0.97	4.40	1.07	4.09	1.14
1/64	9.02	0.05	4.77	0.38	3.17	0.62	2.53	0.80	2.17	0.92

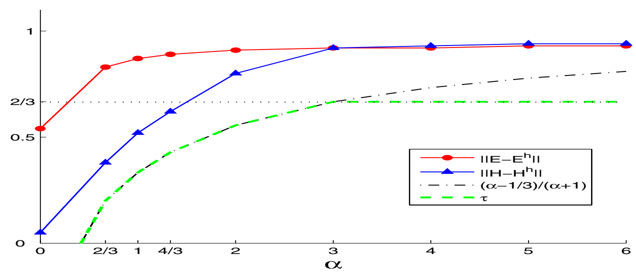

FIG. 7.3. *Finite element convergence rate, τ, as a function of α for Example 7.3.*

TABLE 7.5
AMG convergence factors for Example 7.3.

	$\alpha=0$	$\alpha=2/3$	$\alpha=4/3$	$\alpha=2$	$\alpha=3$	$\alpha=4$	$\alpha=5$	$\alpha=6$
1/8	0.64	0.66	0.66	0.63	0.66	0.66	0.66	0.65
1/16	0.68	0.67	0.68	0.67	0.66	0.67	0.66	0.68
1/32	0.67	0.68	0.68	0.66	0.67	0.66	0.68	0.68
1/64	0.63	0.65	0.65	0.65	0.66	0.67	0.67	0.67

The AMG convergence factors are slightly worse, but still quite acceptable, for discontinuous coefficients, as indicated in Table 7.5. Again, we believe that an improved AMG algorithm may overcome this difficulty.

8. Conclusion. In this paper, we developed a FOSLL* method with a partially weighted norm for the eddy current approximation to Maxwell's equations on a three-dimensional domain with a reentrant edge. We have shown the existence of an H^1-sequence converging to the solution of the eddy current problem in the partially weighted functional norm. This allows accurate approximation using standard H^1-conforming finite element spaces. An L^2-error estimate was established that depends continuously on the weight parameter, α. Numerical tests support our theory. In the future, we will apply our theory to other problems, like full Maxwell's equations, elasticity equations, and Navier–Stokes equations. Also, the reentrant corners (e.g., the Fichera cube) will be considered. We don't anticipate the results, but we believe that our theory can be easily extended to these problems.

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