MULTILEVEL FIRST-ORDER SYSTEM LEAST SQUARES (FOSLS) FOR HELMHOLTZ EQUATIONS

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Abstract. The scalar two-dimensional Helmholtz equation that describes scattering of a radar frequency wave by a body is generally complex and of saddle-point type, making the development of robust and effective discretization and algebraic solution methods problematic. The subject of the present paper is a reformulation of the problem as a least-squares principal for an equivalent augmented first-order system. This recasting of the Helmholtz equation as a well-posed minimization problem enables accurate discretization and optimal multigrid solution methods. This paper develops this methodology and reports on preliminary numerical experiments with model problems.

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1. Introduction

In earlier work [3] [4], we developed least-squares functionals for a first-order system formulation of general second-order elliptic scalar partial differential equations. The functional developed in [4] was shown to be elliptic in the sense that its homogeneous form applied to the $n+1$ variables (*pressure* and *velocities*) is equivalent to the $(H^1)^{n+1}$ norm. This means that the individual variables in the functional are essentially decoupled (more precisely, their interactions are essentially subdominant). This important property ensures that standard finite element methods are of H^1-optimal accuracy in each variable, and that multiplicative and additive multigrid methods applied to the resulting discrete equations are optimally convergent.

The purpose of the present paper is to apply this FOSLS methodology to the (complex and indefinite) Helmholtz equation that models radar wave scattering in two dimensions. The aim is to provide an efficient discretization and multilevel solution process that is robust over a wide range of frequencies, materials, and geometries.

We describe the FOSLS methodology applied to the Helmholtz equation in the next section. This is followed in section 3 by a description of the basic discretization and multigrid solution processes. Section 4 describes a special *ray coarsening* procedure needed in the multigrid method for large frequencies. The paper concludes with a report on model numerical experiments in section 5 and with final remarks in section 6.

2. FOSLS for the Helmholtz Equation

Consider the two-dimensional exterior Helmholtz equation

$$
\begin{aligned}
\nabla \cdot \nabla p + k^2 p &= 0 \qquad \text{in } \Omega^C, \\
p &= f \qquad \text{on } \Gamma, \\
\lim_{r\to\infty} \sqrt{r}(p_r - ikp) &= 0,
\end{aligned} \tag{1}
$$

where $k \in \mathbb{R}$ is the wave number, Ω^C is the complement of some (possibly empty) bounded set in $\mathbb{R}^2$ with boundary Γ, and the expression in the limit argument is considered as a function of polar coordinates where θ is integrated out (Sommerfield radiation condition). We first truncate the infinite domain to some bounded region Ω_R^C with outer boundary a circle of radius R centered on the origin, then replace the Sommerfield radiation condition with

$$
\sqrt{r}(p_r - ikp) = 0 \qquad \text{on } \partial\Omega_R^C/\Gamma. \tag{2}
$$

(Other more accurate approximations to the Sommerfield condition may also be considered; cf. [1] and [2].)

The difficulty with conventional methods for solving the Helmholtz equation is the indefiniteness of the discrete complex system when the wave number is large. Of course, some iterative methods can be applied (e.g., the QMR scheme [5] and other preconditoned conjugate gradient-type schemes [2]). One can also consider the normal equation approach on some discrete approximation to (1). However, most approaches of these types are far from optimal in the sense of delivering an approximation that is accurate to discretization error at a cost proportional to the number of grid points (i.e., the usual sense of multigrid optimality). To achieve such optimality, we will instead convert the scalar problem to a carefully augmented first-order system, then apply a least-squares finite element method ([3], [4], [5], [6]). This will have the effect of formulating a fully elliptic positive definite system of equations in useful physical variables that can be treated by standard Rayleigh-Ritz type finite elements and an associated *multilevel projection method* (cf. [7]).

To develop this FOSLS formulation, we introduce the physical flux vector variable $\boldsymbol{u} = \nabla p$, so that the scalar can be written as the system

$$
\begin{aligned}
\left.\begin{aligned}
\boldsymbol{u} - \nabla p &= \boldsymbol{0} \\
\nabla \cdot \boldsymbol{u} + k^2 p &= 0
\end{aligned}\right\} &\quad \text{in } \Omega_R^C \\
p = f \qquad\quad &\quad \text{on } \Gamma \\
\boldsymbol{u} \cdot \boldsymbol{n} - ikp = 0 \qquad &\quad \text{on } \partial\Omega_R^C/\Gamma,
\end{aligned} \tag{3}
$$

where $\boldsymbol{n}$ is the outward unit normal on $\partial\Omega_R^C/\Gamma$. In operator notation, we have

$$\begin{bmatrix} I & -\nabla \\ \nabla\cdot & k^2 I \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \mathbf{0}. \tag{4}$$

The least-squares system is created formally by multiplying the above equation by the adjoint of the matrix operator:

$$\begin{bmatrix} I & -\nabla \\ \nabla\cdot & k^2 I \end{bmatrix} \begin{bmatrix} I & -\nabla \\ \nabla\cdot & k^2 I \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \mathbf{0}. \tag{5}$$

Unfortunately, components consisting of divergence-free vector fields (e.g., $\boldsymbol{u} \ni \nabla\cdot\boldsymbol{u} = 0$) and zero pressure are eigenfunctions of the operator in (5) with eigenvalue 1, whether these components are smooth or oscillatory. Thus, the operator is not elliptic in the usual H^1 sense. To create this type of ellipticity and enable fast multigrid solution, we first note that, at the solution, the vector flux satisfies the curl condition

$$\nabla\times\boldsymbol{u} = \mathbf{0} \text{ in } \Omega_R^C \tag{6}$$

and the tangential boundary condition

$$\boldsymbol{n}\times\boldsymbol{u} = \boldsymbol{n}\times\nabla f \text{ on } \Gamma. \tag{7}$$

This leads to the augmented system

$$\left.\begin{aligned} \boldsymbol{u} - \nabla p &= \mathbf{0} \\ \nabla\cdot\boldsymbol{u} + k^2 p &= 0 \\ \nabla\times\boldsymbol{u} &= \mathbf{0} \end{aligned}\right\} \text{ in } \Omega_R^C$$
$$\left.\begin{aligned} p &= f \\ \boldsymbol{n}\times\boldsymbol{u} &= \boldsymbol{n}\times\nabla f \end{aligned}\right\} \text{ on } \Gamma \tag{8}$$
$$\boldsymbol{u}\cdot\boldsymbol{n} - ikp = 0 \quad \text{on } \partial\Omega_R^C/\Gamma.$$

The least-squares problem is defined as the minimum of the least-squares functional

$$F(\boldsymbol{u}, p) = \int_{\Omega_R^C} \left\{ |\boldsymbol{u} - \nabla p|^2 + |\nabla\cdot\boldsymbol{u} + k^2 p|^2 + |\nabla\times\boldsymbol{u}|^2 \right\} d\Omega, \tag{9}$$

over an appropriate space satisfying the boundary conditions. For example, such a space is

(10)

$$V = \left\{ (\boldsymbol{u}, p) \in \left[H^1\left(\Omega_R^C\right)\right]^3 \mid p = f \text{ and } \boldsymbol{n}\times\boldsymbol{u} = \boldsymbol{n}\times\nabla f \text{ on } \Gamma,\ \boldsymbol{u}\cdot\boldsymbol{n} - ikp = 0 \text{ on } \partial\Omega_R^C \right\}.$$

3. Numerical Approach

A major advantage of posing the problem as minimizing the least-squares functional is that, through the introduction of a sequence of nested finite element spaces, design of a multigrid solver for the problem is greatly simplified. The algorithm we have in mind here is the multilevel projection method (PML [7]). For this approach, the functional provides a natural goal of computation that, together with the nested spaces, simplifies the choice of relaxation and dictates the definition of other needed multigrid components (interpolation, restriction, and coarse grid operators). This can be especially important with highly indefinite Helmholtz problems, where it can be very difficult to construct suitable coarse grid strategies. The use of PML also has another advantage: it is easy to implement an equivalent *unigrid*, or single-grid, version of the algorithm [8]. Although computationally more expensive than the true multilevel version, the unigrid approach simplifies implementation and testing. Here, we will describe the multilevel projection method for abstract functional minimization and its unigrid counterpart, then develop the extension of the method to our present problem.

One of the differences between having a problem formulated as an equation and as a functional minimization is that, although they have the same solution, the functional provides a natural measure for determining how well a function approximates the solution. (Ideally, this measure should reflect the type of error that is acceptable or unacceptable in the physical application being modelled.) Minimization of the functional can be used as a guiding principle in multilevel projection methods. All processes involved are designed so that the functional is reduced as much as possible in each algorithmic stage.

The resulting algorithm is easy to describe. Suppose we have a minimization problem corresponding to the least-squares solution of an abstract equation $Lu = f$: find $u \in V$ such that

$$F(u) = \min_{v \in V} F(v), \tag{11}$$

where $F(u) = \langle Lu - f, Lu - f\rangle$, V is some suitable space, and $\langle , \rangle$ is some specified inner product. Now the discrete problem is specified by taking some finite-dimensional "fine grid" subspace of V, denoted by V^1, and considering the least-squares solution of the associated overdetermined equation $Lu^1 = f$: find $u^1 \in V^1$ such that

$$F(u^1) = \min_{v^1 \in V^1} F(v^1). \tag{12}$$

This is the target of the numerical solution process. To obtain a multilevel algorithm, we next introduce a nested sequence of auxiliary "coarse grid" subspaces $V^m \subset V^{m-1} \subset \cdots \subset V^1$, and let $B^j = \{b_i^j\}$ be a suitable basis for V^j. Each V^j is usually a finite element space obtained from a uniform coarsening of the element partition used for level $j-1$, and B^j is the standard basis for that space.

Now given an initial solution approximation u, we define a simple level j relaxation sweep as follows:

> **For each $i = 1, 2, \ldots$ in turn, set $u \leftarrow u + \alpha b_i^j$, where α is chosen to minimize $F(u + \alpha b_i^j)$.**

The fact that F is a least-squares formulation of a linear problem means that, for u fixed, F is a quadratic function of α, and the minimization is simple. In fact, this relaxation scheme is equivalent to Gauss-Seidel relaxation on the matrix problem

$$A^j x^j = c^j, \tag{13}$$

where

$$a_{k,l} \;=\; \langle Lb_l^j, Lb_k^j\rangle, \quad c_k^j \;=\; -\langle Lu - f, Lb_k^j\rangle. \tag{14}$$

A complete multilevel cycle (corresponding to what is called a (1,0) V-cycle; here we just use relaxation on the coarsest level V^m) is defined as

For $j = 1, 2, \ldots, m$, do one level j relaxation sweep.

Definition of the subspaces (and choice of bases) and type of relaxation will determine whether the process is efficient. For the cycle to reduce all error quickly, the usual multigrid requirement must hold: any error in V^j must either be effectively reduced by relaxation, or it must approximately lie in V^{j+1}. In practice, this is arranged by properly defining relaxation and/or the subspaces.

This simple description hides many of the usual multigrid processes (such as interpolation, restriction, and forming the coarse grid problem), which are necessary because, for computation, we deal with vectors of coefficients of basis functions on different levels, rather than the spaces themselves. An implementation of the complete multilevel algorithm involves either writing functional evaluation routines or assembling the matrices for all levels, and constructing the interpolation and restriction operators (which are implicitly defined here by the basis functions). This is simplified if we can use the subspaces themselves, or at least subspaces of the fine level coefficient space. This is the idea behind unigrid: rather than formulating coarser level problems, all processing is done on the fine grid, using fine grid representations of the coarse grid basis functions (expressed in terms of the fine grid coefficient space).

To describe the spaces used in discretizing (8), let $T^1 = \{T_i^1\}$ be an element partition of the domain (we generally use rectangular elements), defined by some discretization parameter $h > 0$. In addition, for $j = 2, 3, \ldots, m$, let $T^j = \{T_i^j\}$ be a uniform coarsening of T^{j-1}, with discretization parameter $h_j = 2^{j-1}h$. Define the finite element spaces

$$\begin{aligned} V^j = \Big\{\Big(\boldsymbol{u}^j, p^j\Big) \quad | \quad & \boldsymbol{u}^j = (u^j, v^j),\; u_l^j, v_l^j, p_l^j \in P_r(T_l^j), \\ & p^j = g^j \text{ and } \boldsymbol{n}\times\boldsymbol{u} = \boldsymbol{n}\times\nabla g^j \text{ on } \Gamma,\; \boldsymbol{u}^j\cdot\boldsymbol{n} - ikp = 0 \text{ on } \partial\Omega_R^C\Big\}, \end{aligned} \tag{15}$$

where $P_r(T_l^j)$ is the space of polynomials of degree r over element l, and g^j is defined to be an appropriate approximation to f for $j = 1$ and to be 0 for $j > 1$. (We generally use bilinear polynomials. Note that the same degree of polynomials can be used for u, v, and p, and that there is no restriction on the approximation subspaces since inf-sup inequalities and staggered grids needed for mixed methods are not needed for FOSLS.) Thus, the minimization problem becomes: find $(\boldsymbol{u}^1, p^1) \in V^1$ such that

$$F(\boldsymbol{u}^1, p^1) = \min_{(\boldsymbol{v}^1, q^1) \in V^1} F(\boldsymbol{v}^1, q^1). \tag{16}$$

We take B^j to be the standard nodal basis (hat functions) for the space V^j.

4. Ray Coarsening

Standard multigrid methods generally do not work well for highly indefinite problems, including the scalar Helmholtz equation for large k. The main problem is that usual relaxation methods cannot efficiently reduce error components with wavenumber at or near k. For multigrid to work effectively, a level j error component that cannot be reduced by relaxation must be well approximated by V^{j+1}. For such "bad" components to be well approximated when V^{j+1} consists of piecewise bilinear functions, we must have h_{j+1} small compared to the wavelength (i.e., $2^j hk \ll 1$). This restricts the coarsest grid that can be effectively used. The problem on the coarsest level is solved exactly, so the overall solution process can be prohibitively expensive if the coarse level problem is still large. For large enough k, then, standard multigrid becomes impractical, especially when h is small compared to the domain size and the wavelength is a relatively small multiple of h. Instead of trying to force relaxation to converge well on these components, the approach we take (related to work in [9] and first suggested by Achi Brandt) is to introduce new coarser level subspaces of "ray" functions, chosen so that these "bad" error components can be well approximated.

To be a more specific, consider a component given by $\boldsymbol{u} = \nabla p$, where p is written as a sum of terms of the form $se^{ik(\alpha x+\beta y)}$, with $\alpha^2 + \beta^2 = 1$ and s a smooth function that forces the term to match the boundary conditions. Locally (i.e., ignoring boundaries), this function is a solution to the homogeneous Helmholtz problem, so as an error component it cannot be efficiently reduced by relaxation. However, the exponential part of this term can be well approximated on level j if $h_j k \le \delta \ll 1$ (i.e., if the ratio of the mesh size to the wavelength on that level is sufficiently small). Suppose these error components with wavenumber k are approximated sufficiently well on level $j \le j_c$, but not on the coarser levels $j_c+1, \dots, m$. Let $N^{j_c} : V \rightarrow V^{j_c}$ be the projection onto V^{j_c} such that $N^{j_c}(v)$ is defined to have the same values as v at the level j_c nodes. Thus, by our choice of j_c, $N^{j_c}(se^{ik(\alpha x+\beta y)})$ is a good approximation to $se^{ik(\alpha x+\beta y)}$, but for $j > j_c$ this component cannot be well approximated by any function in V^j. However, if the coefficient function s is sufficiently smooth, then s is well approximated by v^j for some $v^j \in V^j$, so $se^{ik(\alpha x+\beta y)}$ can instead be approximated sufficiently well by the *ray* function $N^{j_c}(v^j e^{ik(\alpha x+\beta y)})$. To use this principle, for fixed α, β and $j > j_c$ we define the ray space

$$V^j_{\alpha,\beta} \;=\; \left\{ d \in V^{j_c} | d = N^{j_c}(v^j e^{ik(\alpha x+\beta y)}), \;\; v^j \in V^j \right\}. \tag{17}$$

A basis for this space is given simply by taking the d's obtained using the basis functions for V^j. Now, for $j > j_c$, instead of using the usual coarse subspace V^j, we use the ray subspaces $V^j_{\alpha,\beta}$.

It is of course infeasible to use all of the infinitude of possible values of (α, β) on each level. Fortunately, only a few are needed. In fact, on the finest level, kh must be small enough to resolve the physics so that rays are usually unnecessary. But at some coarser level (i.e., $j \approx j_c$), slow convergence of standard coarsening begins to be felt and a few (e.g., four) well-separated *representative* rays should be introduced (e.g. $(\alpha, \beta) = (1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$). More rays are unnecessary because all others are well approximated by these representative rays. However, coarser grid scales begin to expose more rays, but a doubling of the representative rays is precisely what is needed for good approximation and for controlling complexity. This implies that the number of grids doubles upon descent to each coarser level; but since the number of corresponding grid points decreases by a factor of about 4, the total multigrid complexity remains proportional to relaxation itself. It is this type of strategy that is used in our experiments reported in the next section.

5. Numerical Results

Here we present the results showing the behavior of our FOSLS methodology on two test problems. For simplicity we avoid the issue of discretization error and instead focus on perhaps the more difficult issue of algebraic convergence factors of the multigrid scheme. The first problem models the electromagnetic field in a nonconducting medium in a square with no body present. The second models the electromagnetic field around an uncoated cylinder. The unigrid approach was used in all tests, and all problems considered were homogeneous, so that accurate asymptotic convergence factors could be obtained.

5.1. Electromagnetic Field in the Square. Our first example is the Helmholtz equation in the unit square:

$$\begin{array}{rll} p_{xx} + p_{yy} + k^2 p = 0 & \text{in } \Omega = [0,1] \times [0,1] \\ \nabla p \cdot \boldsymbol{n} - ikp = 0 & \text{on } \partial\Omega. \end{array} \tag{18}$$

Note that the solution to this problem is obviously $p = 0$, which is appropriate for studying algebraic convergence properties, especially since it avoids machine precision limits to asymptotic error measures. We convert this into the first-order system

$$\begin{array}{rl} \left.\begin{array}{r} \boldsymbol{u} - \nabla p = \boldsymbol{0} \\ \nabla \cdot \boldsymbol{u} + k^2 p = 0 \\ \nabla \times \boldsymbol{u} = \boldsymbol{0} \end{array}\right\} & \text{on } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} - ikp = 0 \quad & \text{on } \partial\Omega, \end{array} \tag{19}$$

which, in turn, is formulated as a least-squares principle based on the functional

$$F(\boldsymbol{u}, p) \;=\; \int_\Omega \left\{ |\boldsymbol{u} - \nabla p|^2 + |\nabla \cdot \boldsymbol{u} + k^2 p|^2 + |\nabla \times \boldsymbol{u}|^2 \right\} d\Omega. \tag{20}$$

A uniform mesh is used, and the functional is minimized over the space of continuous piecewise bilinear functions on that mesh. In the multilevel algorithm, uniform coarser meshes were used, and the coarsest mesh in each case used four elements. Tests were performed without rays, with two rays on level 2, and with four rays on level 2. In the last two cases, the number of rays per level was doubled on each successively coarser grid. Relaxation was simple Gauss-Seidel, sweeping over points in lexicographic order, and at each point relaxing u, then v, then p. When rays were used, at each point the u, v, and p for each ray were relaxed before proceeding to the next point. On the coarsest level, each ray problem in turn was solved exactly. Convergence factors for the PML algorithm described are given in table 1 for $h = 1/8$ and table 2 for $h = 1/16$. Each table contains results for $k = 0, 1, 2, 3, 4, 5$, and for 0, 2, and 4 rays on level 2. To determine the worst-case behavior, the convergence factor given is the maximum over 20 multilevel cycles, and is measured using the least-squares norm. That is, the square root of the functional was used as a norm for measuring error. Entries are omitted from the tables where acceptable convergence factors were achieved with fewer rays.

Note that the convergence factors for a fixed number of level 2 rays generally degrade with k, as expected. The use of rays improves the results. For smaller k, two rays are sufficient, with no additional improvement when k is increased from 2 to 4. However, for larger k, increasing k to 4 is significant. Convergence factors also seem stable when h is reduced from 1/8 to 1/16.

k	0 rays	2 rays	4 rays
0	0.16	—	—
1	0.51	0.31	0.31
2	0.80	0.49	0.49
3	0.89	0.62	0.55
4	0.91	0.71	0.61
5	0.94	0.78	0.69

TABLE 1. Convergence factors for PML on the unit square, $h = 1/8$.

k	0 rays	2 rays	4 rays
0	0.20	—	—
1	0.52	0.33	0.33
2	0.80	0.55	0.48
3	0.89	0.69	0.60
4	0.93	0.75	0.68
5	0.93	0.78	0.73

TABLE 2. Convergence factors for PML on the unit square, $h = 1/16$.

5.2. Electromagnetic Field Around a Cylinder. For computing the field around a cylinder, we write the Helmholtz equations using polar coordinates:

$$
\begin{array}{rl}
p_{rr} + \frac{1}{r}p_r + \frac{1}{r^2}p_{\theta\theta} + k^2 p = 0 & \text{in } \Omega_R^C \\
p = 0 & \text{on } \Gamma \\
\sqrt{r}(p_r - ikp) = 0 & \text{on } \partial\Omega_R^C/\Gamma.
\end{array} \tag{21}
$$

For this case we define

$$
\boldsymbol{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} rp_r - ikrp \\ p_\theta \end{bmatrix}, \tag{22}
$$

and introduce the divergence and curl-type equations, obtaining the first-order system

$$
\begin{array}{rl}
\left.\begin{array}{r}
u - rp_r + ikrp = 0 \\
v - p_\theta = 0 \\
ru_r + v_\theta + ikr(u+p) = 0 \\
u_\theta - rv_r + ikrv = 0
\end{array}\right\} & \text{in } \Omega_R^C \\
\left.\begin{array}{r}
p = 0 \\
v = 0
\end{array}\right\} & \text{on } \Gamma \\
\sqrt{r}(p_r - ikp) = 0 \quad & \text{on } \partial\Omega_R^C/\Gamma.
\end{array} \tag{23}
$$

With these equations, we define the weighted least-squares functional

$$
F(u,v,p) = \tag{24}
$$

$$
\int_{\Omega_R^C} \left\{ |u - rp_r + ikpr|^2 + |v - p_\theta|^2 + |ru_r + v_\theta + ikr(u+p)|^2 + |u_\theta - rv_r + ikrv|^2 \right\} \frac{1}{r}\, dr\, d\theta.
$$

Note that the definition of F exposes another advantage of using a least-squares functional: it guides us in forming the first-order system itself, which should be defined so that the functional is equivalent to some relevant norm on (u,v,p). In the present example, F is essentially $[H_1(\Omega_R^C)]^3$ equivalent, which is important both for the approximation property of the finite element space and the convergence of multigrid.

The functional is minimized over the finite element space

$$
\begin{aligned}
V^1 = \Big\{ (u,v,p) \mid u,v,p \text{ bilinear in } T_j;\ \ u = 0 \text{ on } \Omega_R^C/\Gamma, \\
v = p = 0 \text{ on } \Gamma,\ \ u,v,p \text{ periodic in } \theta \Big\}.
\end{aligned}
$$

In these tests, we used the same number of mesh points in the r and θ directions, and set the outer radius, R, to 2. Radial line relaxation is used in all tests: along each radial line, u, v, and p at all points are relaxed simultaneously. The mesh size is much larger in the θ than the r direction, so this is needed to obtain good smoothing. When rays are used, a full sweep is performed for each ray before proceeding to the next one. The coarsest mesh in each case again uses four elements. Convergence factors for the test runs are given in tables 3 and 4, with $k = 1, \ldots, 5$ and with 0, 2, and 4 rays on level 2. Table 3 shows results for 8 elements in each direction on the fine grid, while table 4 shows results for 16 elements.

6. Concluding Remarks

The FOSLS formulation of the Helmholtz equation and the multigrid algorithm chosen do yield reasonably good algebraic convergence factors, and this is substantially improved by the ray coarsening formulation. We have recently obtained better results through local refinement without the use of rays, which may be needed only as k gets much larger. This will be explored further in a future article.

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k	0 rays	2 rays	4 rays
1	0.31	0.30	—
2	0.35	0.30	—
3	0.47	0.44	0.29
4	0.47	0.39	0.28
5	0.44	0.29	0.28

TABLE 3. Convergence factors for PML on the domain around a cylinder of radius 1 and outer radius $R = 2$, using 8×8 elements.

k	0 rays	2 rays	4 rays
1	0.31	0.31	—
2	0.44	0.31	—
3	0.62	0.31	—
4	0.67	0.54	0.30
5	0.66	0.57	0.35

TABLE 4. Convergence factors for PML on the domain around a cylinder of radius 1 and outer radius $R = 2$, using 16×16 elements.