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# Improved Discretization Error Estimates for First-Order System Least Squares (FOSLS)

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**Abstract** — We study the discretization accuracy for first-order system least squares (FOSLS) applied to Poisson's equation as a model problem. The FOSLS formulation is based on an $H^1$ elliptic bilinear form $\mathscr{F}$. Since the order of convergence of the discretization in the $L^2$ and $H^1$ norms depends on the regularity of $\mathscr{F}$, we examine this property in detail. We then use these results together with an Aubin-Nitsche bound to develop improved discretization error estimates.

**Keywords:** Least squares discretization, finite elements, error analysis

## 1. INTRODUCTION

First-order system least squares (FOSLS) is an important tool in the numerical solution of partial differential equations (see [2], [3], and the references cited therein). The basic idea behind FOSLS is to write the partial differential equation as a first-order system and apply a least-squares principle, then minimize the resulting functional by Rayleigh-Ritz. Careful reformulation of the original equation as an appropriate first-order system leads to an $H^1$ elliptic bilinear form. This is important because it means that the discretization process attains the usual order of accuracy for the original *and* newly defined variables. ($H^1$ ellipticity also implies optimality of the multigrid solver.)

The purpose of this paper is to establish bounds on discretization accuracy that are stronger than the usual ones. Our focus will be on Poisson's equation with Dirichlet boundary conditions. Although one could study more general equations, the analysis of the regularity that we need is already rather involved and, in any case, it illustrates the techniques that can be used for more general equations.

The classical Galerkin discretization of Poisson's equation using linear finite elements converges with order $h$ in the $H^1$ norm when the solution is in $H^2$. On the other hand, straightforward theory for FOSLS shows that its discretization converges with order $h^\varepsilon$ in the $H^1$ norm when the solution is in $H^{2+\varepsilon}$. However, this estimate for FOSLS holds not only for the original variable $p$, but also for the new

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flux variable $\mathbf{u} := \nabla p$:

$$\|\mathbf{u}-\mathbf{u}_h\|_1 + \|p-p_h\|_1 \leqslant C_{int} h^{\varepsilon} \|p\|_{2+\varepsilon}.$$

In what follows, we show that the order of convergence improves in the $L^2$ norm. With the help of an Aubin-Nitsche bound, we prove that the flux converges with order $h^{\alpha-1+\varepsilon}$ when the bilinear form $\mathscr{F}$ corresponding to the FOSLS discretization is $H^{\alpha}$ regular (see Definition 3.1 and Theorem 4.1):

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \leqslant C_{int} h^{\alpha-1+\varepsilon} \|p\|_{2+\varepsilon}.$$

For $\varepsilon = 0$ and $\alpha = 2$, this result is comparable to that for the Galerkin discretization, which requires that the solution of Poisson's equation be in $H^2(\Omega)$ and concludes that the discretized flux converges with order $h$ in the $L^2$ norm. This shows that it is necessary to perform a careful analysis of the regularity of the FOSLS bilinear form $\mathscr{F}$ (see section 3). We are, in fact, able to prove $H^2$ regularity of $\mathscr{F}$ for the square and for domains with smooth boundaries. For convex polygonal domains, our regularity estimates depend on the largest interior angle (see Theorem 3.1).

In the next section, we introduce the problem, FOSLS formulation, and a simple but important lemma. In Section 3, we study regularity of the FOSLS bilinear form. In Section 4, we obtain improved discretization error estimates for the FOSLS formulation.

## 2. FOSLS FORMULATION

Let $\Omega$ be a bounded domain in $\mathbf{R}^2$, with piecewise $\mathscr{C}^2$ boundary $\Gamma = \partial\Omega$ with no reentrant corner. Consider Poisson's equation

$$\begin{aligned} -\triangle p &= f \quad \text{on } \Omega, && (2.1)\\ p &= 0 \quad \text{on } \Gamma, && (2.2) \end{aligned}$$

for $f \in L^2(\Omega)$. Here we assume homogeneous Dirichlet conditions, but it is clear that our results hold for the case $p|_{\Gamma} = g$ on $\Gamma$, $g \in H^{\frac{3}{2}}(\Gamma)$.

We use the following spaces to define a FOSLS functional for (2.1) - (2.2):

$$\begin{aligned} \mathbf{W} &:= \{\mathbf{v} \in (H^1(\Omega))^2 \mid \mathbf{n} \times \mathbf{v} = \mathbf{0} \text{ on } \Gamma\},\\ V &:= H_0^1(\Omega), \end{aligned}$$

where $\mathbf{n}$ is the outward unit normal vector. If $p$ is the unique solution of (2.1) - (2.2), then certainly $p \in V$ and the flux variable

$$\mathbf{u} := \nabla p \qquad (2.3)$$

is in $\mathbf{W}$. It can, in fact, be shown (cf. [3]) that any pair

$$(\mathbf{u}, p) \in \mathbf{W} \times V$$

that satisfies (2.1) - (2.3) must be a minimizer of the least-squares functional

$$\mathscr{G}(\mathbf{u},p;f) := \|\nabla p - \mathbf{u}\|_{0,\Omega,2}^2 + \|\nabla\cdot\mathbf{u} + f\|_{0,\Omega}^2 + \|\nabla\times\mathbf{u}\|_{0,\Omega,2}^2,$$

and conversely. Here, $\|\cdot\|_{0,\Omega}$ denotes the $L^2(\Omega)$ norm (with inner product $(\cdot,\cdot)_{0,\Omega}$) and $\|\cdot\|_{0,\Omega,m}$ the $(L^2(\Omega))^m$ graph norm (with inner product $(\cdot,\cdot)_{0,\Omega,m}$). The least-squares bilinear form associated with this functional is

$$\begin{aligned}&\mathscr{F}(\mathbf{u},p;\mathbf{v},q)\\ &:= \quad (\nabla p - \mathbf{u}, \nabla q - \mathbf{v})_{0,\Omega,2} + (\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v})_{0,\Omega} + (\nabla\times\mathbf{u}, \nabla\times\mathbf{v})_{0,\Omega,2}.\end{aligned}$$

It is evident from the Cauchy-Schwarz inequality that a constant $\beta$ exists such that

$$|\mathscr{F}(\mathbf{w},r;\mathbf{v},q)| \leqslant \beta\|(\mathbf{w},r)\|_{1,\Omega,3}\|(\mathbf{v},q)\|_{1,\Omega,3} \quad \forall(\mathbf{w},r),(\mathbf{v},q) \in \mathbf{W}\times V, \tag{2.4}$$

where $\|\cdot\|_{1,\Omega,3}$ represents the usual $(H^1(\Omega))^3$ graph norm. Defining

$$F(\mathbf{v},q) := -(f,\nabla\cdot\mathbf{v})_{0,\Omega},$$

then the minimizer $(\mathbf{u},p) \in \mathbf{W}\times V$ of $\mathscr{G}(\mathbf{u},p;f)$ is the unique solution of the weak equation

$$\mathscr{F}(\mathbf{u},p;\mathbf{v},q) = F(\mathbf{v},q) \qquad \forall(\mathbf{v},q) \in \mathbf{W}\times V. \tag{2.5}$$

The bilinear function $\mathscr{F}$ is defined on vector variables (i.e., $(\mathbf{u},p)$ and $(\mathbf{v},q)$ are 3-vectors). The system of equations arising from this bilinear form (i.e., the Euler or gradient equation for minimizing $\mathscr{F}$) can be separated into individual equations in several ways. A very simple separation is used to prove the following lemma, which will be useful in subsequent sections.

**Lemma 2.1 Reduction of the System of Equations.** *Assume that $G \in (\mathbf{W}\times V)'$ and that $(\mathbf{w},r) \in \mathbf{W}\times V$ such that*

$$\mathscr{F}(\mathbf{w},r;\mathbf{v},q) = G(\mathbf{v},q) \qquad \forall(\mathbf{v},q) \in \mathbf{W}\times V.$$

*Then*

$$(\nabla r, \nabla q)_{0,\Omega,2} = (\mathbf{w},\nabla q)_{0,\Omega,2} + G(\mathbf{0},q) \qquad \forall q \in V$$

*and*

$$\begin{aligned}(\nabla\cdot\mathbf{w},\nabla\cdot\mathbf{v})_{0,\Omega} + (\nabla\times\mathbf{w},\nabla\times\mathbf{v})_{0,\Omega,2} \quad &= \quad (\nabla r - \mathbf{w},\mathbf{v})_{0,\Omega,2} + G(\mathbf{v},0)\\ &\qquad\qquad \forall\mathbf{v} \in \mathbf{W}.\end{aligned}$$

**Proof.** This follows immediately by applying the test functions $(\mathbf{0},q)$ and $(\mathbf{v},0)$ to (2.5). □

## 3. REGULARITY RESULTS

Consider the following concept of regularity.

**Definition 3.1** **$H^\alpha$ regularity of $\mathscr{F}$.** Assume that $\alpha \geqslant 1$. The bilinear form $\mathscr{F}$ is $H^\alpha$ regular if there exists a constant $C_{reg}$ such that:

> For every $g = (g_1, g_2, g_3) \in (L^2(\Omega))^3$, the solution $(\mathbf{w}, r) \in \mathbf{W} \times V$ of the weak equation
>
> $$\mathscr{F}(\mathbf{w}, r; \mathbf{v}, q) = (g; (v, q))_{0,\Omega,3} \qquad \forall (\mathbf{v}, q) \in \mathbf{W} \times V$$
>
> is contained in $(H^\alpha(\Omega))^3$ and satisfies
>
> $$\|(\mathbf{w}, r)\|_{\alpha,\Omega,3} \leqslant C_{reg} \|g\|_{0,\Omega,3}.$$

We want to prove $H^\alpha$ regularity of the bilinear form $\mathscr{F}$. For this, we study the following three cases of $\Omega$:

*Case 1*. The domain $\Omega$ is the square $]0,1[^2$.
*Case 2*. The boundary $\Gamma$ is $\mathscr{C}^2$.
*Case 3*. The domain $\Omega$ is a polygon where interior angles are less than $\frac{2}{3}\pi$.

Our results can also be extended to domains that are locally isomorphic to one of these cases, but we omit this possibility for simplicity.

Before we can analyze the regularity of $\mathscr{F}$, we must study regularity of the Div-Curl bilinear form

$$\mathscr{D}(\mathbf{w}, \mathbf{v}) := (\nabla\cdot\mathbf{w}, \nabla\cdot\mathbf{v})_{0,\Omega,2} + (\nabla\times\mathbf{w}, \nabla\times\mathbf{v})_{0,\Omega,2}.$$

**Definition 3.2** **$H^\alpha$ regularity of Div-Curl.** Assume that $\alpha \geqslant 1$. The Div-Curl bilinear form $\mathscr{D}$ is $H^\alpha$ regular if there exists a constant $C^\alpha_{reg}$ such that:

> For every $\mathbf{g} = (g_1, g_2) \in (L^2(\Omega))^2$, the solution $\mathbf{w} \in \mathbf{W}$ of the weak equation
>
> $$\mathscr{D}(\mathbf{w}, \mathbf{v}) = (\mathbf{g}, \mathbf{v})_{0,\Omega,2} \quad \forall \mathbf{v} \in \mathbf{W} \tag{3.1}$$
>
> is contained in $(H^\alpha(\Omega))^2$ and satisfies
>
> $$\|\mathbf{w}\|_{\alpha,\Omega,2} \leqslant C^\alpha_{reg} \|\mathbf{g}\|_{0,\Omega,2}.$$

Our next lemma introduces a strong formulation whose $(H^1(\Omega))^2$ solution, if it exists, must satisfy the Div-Curl weak form. This strong formulation requires the existence of a trace operator. Obviously, such an operator exists if the solution is in $(H^{\alpha}(\Omega))^2$ with $\alpha > 1.5$. One can show that a suitable trace operator also exists for the case $\alpha \geqslant 1$, since the Laplacian of the solution is contained in $L^2$. However, the proof of existence of such an operator is rather technical, so we instead restrict ourselves to the case $\alpha > 1.5$.

**Lemma 3.1.** *Assume that* $\mathbf{g} = (g_1, g_2) \in L^2(\Omega)^2$ *and* $\alpha > 1.5$. *If the system*

$$\begin{aligned} -\triangle \mathbf{w} &= \mathbf{g} \quad \text{on } \Omega, \\ \mathbf{n} \times \mathbf{w} &= \mathbf{0} \quad \text{on } \Gamma, \\ \nabla\cdot\mathbf{w} &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.2}$$

*has a solution* $\mathbf{w} \in (H^{\alpha}(\Omega))^2$, *then it must also be the solution in* $\mathbf{W} \cap (H^{\alpha}(\Omega))^2$ *of the weak form*

$$\mathscr{D}(\mathbf{w}, \mathbf{v}) = (\mathbf{g}, \mathbf{v})_{0,\Omega,2} \quad \forall \mathbf{v} \in \mathbf{W}.$$

**Proof.** Define

$$E = \left\{ \mathbf{w} \in (H^{\alpha}(\Omega))^2 \mid \triangle \mathbf{w} \in (L^2(\Omega))^2 \right\}$$

with norm $\|\cdot\|_E = \sqrt{\|\mathbf{w}\|^2_{\alpha,\Omega,2} + \|\triangle \mathbf{w}\|_{0,\Omega,2}}$. Then a proof similar to that of Theorem 3.6 in [10] shows that $\mathscr{C}^2(\bar{\Omega}) \cap E$ is dense in $E$. Let $\mathbf{w} \in \mathbf{W} \cap E$ be such that $\nabla\cdot\mathbf{w} = 0$ on $\Gamma$ and suppose that $\mathbf{w}^n \in E \cap \mathscr{C}^2(\bar{\Omega})$ is a sequence that converges to $\mathbf{w}$ with respect to the norm $\|\cdot\|_E$. Thus, $\triangle \mathbf{w}_n$ converges to $\triangle \mathbf{w}$ in the $L^2(\Omega)$ norm and, because $\alpha > 1.5$, $\nabla\cdot\mathbf{w}_n$ certainly converges to $\nabla\cdot\mathbf{w} = 0$ in the $L^2(\Gamma)$ norm. Assume $\mathbf{v} \in \mathbf{W}$. Then, by the Gauss Divergence Theorem and the fact that $\mathbf{n} \times \mathbf{v} = \mathbf{0}$ on $\Gamma$, we have

$$\begin{aligned} &(\nabla\cdot\mathbf{w}^n, \nabla\cdot\mathbf{v})_{0,\Omega,2} + (\nabla\times\mathbf{w}^n, \nabla\times\mathbf{v})_{0,\Omega,2} \\ &= -\int_{\Omega} \triangle w_1^n v_1 + \triangle w_2^n v_2 \, d\Omega + \int_{\Gamma} (\nabla\cdot\mathbf{w}^n)\,(\mathbf{v}\cdot\mathbf{n})\, d\Gamma. \end{aligned}$$

Taking the limit as $n \to \infty$ leads to

$$\begin{aligned} (\nabla\cdot\mathbf{w}, \nabla\cdot\mathbf{v})_{0,\Omega,2} + (\nabla\times\mathbf{w}, \nabla\times\mathbf{v})_{0,\Omega,2} &= -\int_{\Omega} \triangle \mathbf{w}\, \mathbf{v}\, d\Omega + \int_{\Gamma} (\nabla\cdot\mathbf{w})\,(\mathbf{v}\cdot\mathbf{n})\, d\Gamma \\ &= -\int_{\Omega} \triangle \mathbf{w}\, \mathbf{v}\, d\Omega. \end{aligned}$$

□

For a polygon domain, there are other useful boundary conditions.

**Lemma 3.2.** *Assume that $\Omega$ is a polygon domain. Then the boundary conditions*

$$\begin{aligned} \mathbf{n} \times \mathbf{w} &= \mathbf{0} \quad \text{on } \Gamma, \\ \nabla \cdot \mathbf{w} &= 0 \quad \text{on } \Gamma, \end{aligned}$$

*are equivalent to*

$$\begin{aligned} \mathbf{n} \times \mathbf{w} &= \mathbf{0} \quad \text{on } \Gamma, \\ \mathbf{n} \circ \frac{\partial \mathbf{w}}{\partial \mathbf{n}} &= 0 \quad \text{on } \Gamma. \end{aligned}$$

**Proof.** By a short calculation, on $\Gamma$ we obtain

$$\begin{aligned} \mathbf{n} \circ \frac{\partial \mathbf{w}}{\partial \mathbf{n}} &= \nabla \cdot \mathbf{w} - \mathbf{n} \times \frac{\partial \mathbf{w}}{\partial \tau} \\ &= \nabla \cdot \mathbf{w} - \frac{\partial (\mathbf{n} \times \mathbf{w})}{\partial \tau}, \end{aligned}$$

where $\tau$ is the tangent vector. Since $\mathbf{n} \times \mathbf{w} = 0$ on $\Gamma$, then $\frac{\partial (\mathbf{n} \times \mathbf{w})}{\partial \tau} = 0$, which completes the proof. □

We now come to the main result of this section.

**Theorem 3.1.** *Let $\omega$ be the size of the largest interior angle of $\Omega$. If $\omega > \frac{\pi}{2}$, then $\mathscr{F}$ is $H^{\frac{\pi}{\omega}-\varepsilon}$ regular for any fixed $\varepsilon \in (0, \frac{\pi}{\omega} - 1]$. Otherwise, $\mathscr{F}$ is $H^2$ regular.*

**Proof.** We have to prove that $\mathscr{F}$ is $H^\alpha$ regular, where we choose $\alpha$ as follows:
*Case 1.* Let $\alpha = 2$.
*Case 2.* Let $\alpha = 2$.
*Case 3.* Let $\alpha = 2$, if $\omega \leqslant \frac{\pi}{2}$. Otherwise, let $\alpha = \frac{\pi}{\omega} - \varepsilon$ for any fixed $\varepsilon \in (0, \frac{\pi}{\omega} - 1]$.

Suppose that we could prove that, for every $\mathbf{g} = (g_1, g_2) \in (L^2(\Omega))^2$, the solution $\mathbf{w}$ of the system

$$-\triangle \mathbf{w} = \mathbf{g} \quad \text{on } \Omega, \tag{3.3}$$
$$\mathbf{n} \times \mathbf{w} = \mathbf{0} \quad \text{on } \Gamma, \text{ and} \tag{3.4}$$
$$\nabla \cdot \mathbf{w} = 0 \quad \text{on } \Gamma \tag{3.5}$$

is in $(H^\alpha(\Omega))^2$. Then, by Lemma 3.1, the Div-Curl bilinear form is $H^\alpha$ regular. It is well known that Poisson's equation (2.1) - (2.2) is $H^2$ regular since the domain has no reentrant corners (see [5]). But Lemma 2.1 shows that regularity of the bilinear form $\mathscr{F}$ follows from regularity of Poisson's equation and the Div-Curl bilinear form. We could thus conclude that the bilinear form $\mathscr{F}$ is $H^\alpha$ regular.

To prove that the solution $\mathbf{w}$ of (3.3) - (3.5) is in $(H^\alpha(\Omega))^2$, we consider the three cases individually.

*Case 1.* The boundary $\Gamma$ consists only of horizontal $\Gamma_{\text{hor}}$ and vertical $\Gamma_{\text{ver}}$ lines, and boundary conditions (3.4) and (3.5) lead to

$$w_2 = 0 \quad \text{and} \quad \frac{\partial w_1}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_{\text{ver}},$$
$$w_1 = 0 \quad \text{and} \quad \frac{\partial w_2}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_{\text{hor}}.$$

Equation (3.3) becomes

$$\begin{aligned} -\triangle w_1 &= g_1 \quad \text{on } \Omega, \\ -\triangle w_2 &= g_2 \quad \text{on } \Omega. \end{aligned}$$

This shows that (3.3) - (3.5) is a decoupled system of two Poisson equations with mixed Dirichlet and Neumann boundary conditions on the square. These equations are known to be $H^2$ regular (see [5]). We thus have $\mathbf{w} \in (H^{\alpha}(\Omega))^2$, as desired.

*Case 2.* As the basis for a continuation argument, consider the one-parameter family of operators $(L_t, B_t) : X_M \to Y_M \times Z_M$ defined by

$$(L_t, B_t) : \mathbf{w} = (w_1, w_2)^t \mapsto \begin{pmatrix} \triangle \mathbf{w} \\ \mathbf{n} \times \mathbf{w}\, t + (1-t)\, w_1 \\ \nabla \cdot \mathbf{w}\, t + (1-t)\, \frac{\partial w_2}{\partial \mathbf{n}} \end{pmatrix},$$

where $t \in [0,1]$ and

$$\begin{aligned} X_M &= (H^2(\Omega))^2, \\ Y_M &= (L^2(\Omega))^2, \\ Z_M &= H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega). \end{aligned}$$

Using ADN theory (see [1] ), we will show that there exits a constant $C$, which depends only on $t$, such that

$$\|(w_1, w_2)\|_{X_M} \leqslant C\Big( \|L_t(w_1, w_2)\|_{Y_M} + \|B_t(w_1, w_2)\|_{Z_M} + \|(w_1, w_2)\|_{0,\Omega,2} \Big) \qquad (3.6)$$

for every $(w_1, w_2) \in X_M$, and that, if $\dim(\text{kern}(L_t, B_t)) = 0$, then $C$ can be chosen such that

$$\|(w_1, w_2)\|_{X_M} \leqslant C\Big( \|L_t(w_1, w_2)\|_{Y_M} + \|B_t(w_1, w_2)\|_{Z_M} \Big) \qquad (3.7)$$

for every $(w_1, w_2) \in X_M$. Before we prove this result, consider its consequences. By Corollary 8.32 in [9], $(L_t, B_t)$ are semi-Fredholm operators. Obviously, the mapping

$$[0,1] \to \mathscr{L}(X_M, Y_M \times Z_M)$$

defined by

$$t \mapsto (L_t, B_t)$$

is continuous. Furthermore, $[0,1]$ is connected. Thus, the set

$$\Big\{(L_t, B_t) \mid t \in [0,1]\Big\}$$

is a connected set of semi-Fredholm operators with respect to the topology induced by the operator norm on $\mathscr{L}(X_M, Y_M \times Z_M)$. By Corollary 7.103 in [9], the indices of $(L_0, B_0)$ and $(L_1, B_1)$ must be equal. Now, the index of the Poisson equation with Dirichlet boundary conditions is zero. The index for Poisson's equation with Neumann conditions is also zero because kern and cokern consist of the constant functions. This implies that the index of $(L_0, B_0)$ is zero, which in turn implies that the index of $(L_1, B_1)$ is zero. By the definition of index, we obtain

$$\dim(\mathrm{kern}(L_1, B_1)) = \dim(\mathrm{cokern}(L_1, B_1)). \tag{3.8}$$

Now, $H^2$ regularity of $(L_1, B_1)$ would follow if we could establish both (3.7) and the additional property that $\dim(\mathrm{kern}(L_1, B_1)) = 0$, that is, $H^2$ regularity would be proved if we could show that

$$\dim(\mathrm{cokern}(L_1, B_1)) = 0 \quad \text{and} \quad \dim(\mathrm{kern}(L_1, B_1)) = 0.$$

By (3.8), it is enough to show that

$$\dim(\mathrm{kern}(L_1, B_1)) = 0.$$

To prove this, assume that $(L_1, B_1)(\mathbf{w}) = 0$ for some $\mathbf{w} \in (H^2(\Omega))^2$. By Lemma 3.1, $\mathbf{w}$ is the solution of the weak equation

$$\mathscr{D}(\mathbf{w}, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{W}.$$

But $\mathscr{D}^{\frac{1}{2}}(\mathbf{w}, \mathbf{w})$ is bounded and coercive (in the $(H^1(\Omega)$ norm; see [3]), which implies that $\mathbf{w} = \mathbf{0}$. Thus, we have proved that $\dim(\mathrm{kern}(L_1, B_1)) = 0$.

Hence, to complete the proof of *Case 2*, we need only establish inequalities (3.6) and (3.7). To this end, let $t \in [0,1]$ be fixed and define

$$\begin{aligned} s_1 = s_2 &= 0, \\ t_1 = t_2 &= 2, \\ r_1 = -2, &\quad r_2 = -1, \\ M = M_1 &= 0. \end{aligned}$$

With these choices, the operator $L_t$ is elliptic of order 2 in the sense of ADN theory, because $L_t$ consists of two Poisson operators (cf. [1]). It remains only to show that the so-called complementing boundary condition of the ADN theory is satisfied.

The complementing boundary condition could be characterized in an algebraic way (see [1]), but here we instead choose an approach developed in [9]. To explain this characterization, for a fixed point $\xi_0 \in \Gamma$, we define a system of differential equations on the half space

$$\chi := \{\xi \in \mathbf{R}^2 \mid (\xi - \xi_0) \circ \mathbf{n}(\xi_0) < 0\},$$

using the coefficients of the operators $L_t$ and $B_t$ at the point $\xi_0$. In our case, the system is

$$\begin{aligned} \triangle \mathbf{w} &= 0 \quad \text{on } \chi, \\ \mathbf{w} \times \mathbf{n}(\xi_0)\, t + (1-t)\, w_1 &= 0 \quad \text{on } \partial\chi, \\ \nabla\cdot\mathbf{w}\, t + (1-t)\, \frac{\partial w_2}{\partial \mathbf{n}(\xi_0)} &= 0 \quad \text{on } \partial\chi, \end{aligned}$$

where $\mathbf{w} \in (H^2(\chi))^2$. For its solution, we make the following ansatz:

$$\mathbf{w}(\xi) = \exp\Big(i\mathbf{n}(\xi_0) \times (\xi - \xi_0)\Big) \exp\Big(\mathbf{n}(\xi_0) \circ (\xi - \xi_0)\lambda\Big) \mathbf{v}_0,$$

where $\mathbf{v}_0 \in \mathbf{R}^2$ and $Re(\lambda) > 0$. Now, we say that the complementing boundary condition is satisfied at the point $\xi_0$ if we can conclude that $\mathbf{v}_0$ must be zero.

To show that this is indeed true in our case, we first transform the coordinate system so that $\xi_0 = (0,0)$. With the abbreviation $\mathbf{n} = \mathbf{n}(\xi_0)$, then

$$\chi := \{\xi \in \mathbf{R}^2 \mid \xi \circ \mathbf{n} < 0\},$$

and we obtain the following system of differential equations:

$$\begin{aligned} \triangle \mathbf{w} &= \mathbf{0} \quad \text{on } \chi, \\ \Big(\mathbf{w} \times \mathbf{n}\, t + (1-t)\, w_1\Big) &= 0 \quad \text{on } \partial\chi, \\ \Big(\nabla\cdot\mathbf{w}\, t + (1-t)\, \frac{\partial w_2}{\partial \mathbf{n}}\Big) &= 0 \quad \text{on } \partial\chi. \end{aligned}$$

The ansatz becomes

$$\mathbf{w}(\xi) = \exp\Big(i\mathbf{n} \times \xi\Big) \exp\Big(\mathbf{n} \circ \xi\lambda\Big) \mathbf{v}_0.$$

Assume that $\mathbf{v}_0 = (v_1, v_2)^t$ is not zero. Then $\triangle \mathbf{w} = \mathbf{0}$ implies that $\lambda = \pm 1$. By $Re(\lambda) > 0$, we get

$$\lambda = 1.$$

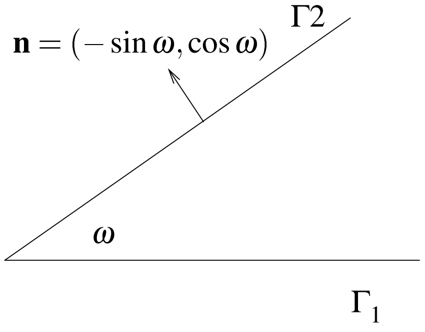


**Figure 1.** Conical domain

The boundary conditions now imply that

$$\begin{aligned} \mathbf{n} \times \mathbf{v}_0 t + (1-t) v_1 &= 0, \\ (\mathbf{n} \times \mathbf{v}_0 i + \mathbf{n} \circ \mathbf{v}_0) t + (1-t) v_2 &= 0. \end{aligned}$$

A simple calculation then shows that $\mathbf{v}_0 = \mathbf{0}$. We thus conclude that the complementing boundary condition is satisfied, and we are now able to apply Theorem 8.31 in [9] to obtain the inequalities (3.6) and (3.7). This completes the proof of $H^2$ regularity of (3.3) - (3.5) for *Case 2*.

*Case 3.* By a localization of the differential equation (see [10], section 20, or [4]), we may without loss of generality assume that the domain is a cone:

$$\Omega = \left\{ r(\cos(\varphi), \sin(\varphi)) \mid 0 < r, 0 < \varphi < \omega \right\}.$$

Figure 1 depicts this cone with angle $\omega$ and boundary segments $\Gamma_1$ and $\Gamma_2$, where we note that $\Gamma_1$ lies along the x-coordinate and the normal vector on $\Gamma_1$ is $\mathbf{n}|_{\Gamma_1} = (0, -1)$. By Lemma 3.2, the boundary conditions in (3.2) become

$$\begin{aligned} w_1 &= 0 \quad \text{on } \Gamma_1, \\ \frac{\partial w_2}{\partial y} &= 0 \quad \text{on } \Gamma_1, \\ \mathbf{n} \times \mathbf{w} &= 0 \quad \text{on } \Gamma_2, \\ \mathbf{n} \circ \frac{\partial \mathbf{w}}{\partial \mathbf{n}} &= 0 \quad \text{on } \Gamma_2, \end{aligned} \tag{3.9}$$

and the equation is

$$-\triangle \mathbf{w} = \mathbf{g}. \tag{3.10}$$

We want to determine the regularity of (3.9) - (3.10). By the analysis in *Case 2*, we know that the complementing boundary condition of the ADN theory must be satisfied.

To calculate the singularities in the solution of (3.9) - (3.10), we apply Kondrat'ev's method. This method is described in [6] for scalar equations. But it can easily be extended to systems (see section 3.2.5. in [8] or (7.16) in [4]). Using the notation in [6], let $\tilde{D} = ]0,\omega[$. According section 1 in [6], let us apply polar coordinates $(r,\varphi)$, a change of variables $\tau = \ln\frac{1}{r}$, and a Fourier transformation with respect to $\tau$ to (3.9) - (3.10). This leads to

$$\begin{aligned}
\lambda^2\tilde{w}_1 - \frac{\partial^2}{\partial\varphi^2}\tilde{w}_1 &= \tilde{G}_1 && (3.11)\\
\lambda^2\tilde{w}_2 - \frac{\partial^2}{\partial\varphi^2}\tilde{w}_2 &= \tilde{G}_2 && (3.12)\\
\tilde{w}_1(0) &= 0 && (3.13)\\
\frac{\partial\tilde{w}_2}{\partial\varphi}(0) &= 0 && (3.14)\\
n_1\tilde{w}_2(\omega) - n_2\tilde{w}_1(\omega) &= 0 && (3.15)\\
n_1\frac{\partial\tilde{w}_1}{\partial\varphi}(\omega) + n_2\frac{\partial\tilde{w}_2}{\partial\varphi}(\omega) &= 0 && (3.16)
\end{aligned}$$

where $\tilde{w}_1, \tilde{w}_2$ is the Fourier transformation of $\mathbf{w}$ with respect to $\tau$ and $\tilde{G}_1$, $\tilde{G}_2$ are the transformed right hand sides. Let $\tilde{W}_0^2(\tilde{D})$ be the space of function $(\tilde{w}_1,\tilde{w}_2) \in (H^2(\tilde{D}))^2$ satisfying the boundary conditions (3.13) - (3.16). Let $A(\lambda) : \tilde{W}_0^2(\tilde{D}) \to (L^2(\tilde{D}))^2$ be the operator corresponding to the left hand side of (3.11) - (3.12). There exists a meromorphic operator

$$R(\lambda) : (L^2(\tilde{D}))^2 \to \tilde{W}_0^2(\tilde{D})$$

such that

$$A(\lambda)R(\lambda) = E.$$

To calculate the poles of $R(\lambda)$ for $Im(\lambda) > 0$, one hast to study the kernel of the operator $A(\lambda)$. This means one has to solve the eigenvalue problem

$$\lambda^2(\tilde{w}_1,\tilde{w}_2) = \frac{\partial^2}{\partial\varphi^2}(\tilde{w}_1,\tilde{w}_2), \qquad (3.17)$$

where $(\tilde{w}_1,\tilde{w}_2) \in \tilde{W}_0^2(\tilde{D})$. (3.17) is an ODE for every fixed $\lambda \in \mathbb{C}$. A nontrivial solution of (3.17) for $Im(\lambda) > 0$ exists, if

$$\begin{aligned}
\lambda &= ik,\\
\tilde{w}_1 &= a_1\sin(k\varphi) + b_1\cos(k\varphi),\\
\tilde{w}_2 &= a_2\sin(k\varphi) + b_2\cos(k\varphi),
\end{aligned}$$

where $k>0$ has to be chosen such that the boundary conditions (3.13) - (3.16) are satisfied for $(\tilde{w}_1,\tilde{w}_2)$. A short calculation shows that $A(\lambda)$ is selfadjoint. Thus the multiplicity of the pole of $R(\lambda)$ at $\lambda = ik$ is one and the singularities which may appear in the solution of (3.9) - (3.10) have the following form

$$\begin{aligned} w_1 &= a_1 r^k \sin(k\varphi) + b_1 r^k \cos(k\varphi), \\ w_2 &= a_2 r^k \sin(k\varphi) + b_2 r^k \cos(k\varphi). \end{aligned}$$

This follows by Theorem 4.4. in [6] (see also section 3.2.5. in [8] or (5.11) and (7.16) in [4]).

We have to find the smallest $k$ satisfying the boundary conditions (3.13) - (3.16). By the boundary condition at $\varphi = 0$, it follows

$$\begin{aligned} \tilde{w}_1 &= a\sin(k\varphi), \\ \tilde{w}_2 &= b\cos(k\varphi), \end{aligned}$$

where $a$, $b$, and $k>0$ are real numbers.

Let $(n_x,n_y) = \mathbf{n}|_{\Gamma_2}$ be the outward unit normal vector on $\Gamma_2$. The boundary conditions on $\varphi = \omega$ imply that

$$\begin{aligned} a\sin(k\omega)\,n_y - b\cos(k\omega)\,n_x &= 0, \qquad (3.18) \\ a\cos(k\omega)\,n_x - b\sin(k\omega)\,n_y &= 0. \qquad (3.19) \end{aligned}$$

This equation system has a nontrivial solution $(a,b)$ if and only if

$$\det\begin{pmatrix} \sin(k\omega)\,n_y & -\cos(k\omega)\,n_x \\ \cos(k\omega)\,n_x & -\sin(k\omega)\,n_y \end{pmatrix} = 0,$$

that is,

$$\sin^2(k\omega)n_y^2 - \cos^2(k\omega)n_x^2 = 0.$$

Since $\omega \in (0,\frac{2\pi}{3})$, then $n_x \neq 0$. If $\sin^2(k\omega) = 0$, then we must also have $\cos^2(k\omega) = 0$, which is impossible. Thus, we must have $\sin^2(k\omega) \neq 0$. Writing $(n_x,n_y) = (-\sin(\omega),\cos(\omega))$ (see Figure 1), then we must have

$$\cot^2(k\omega) = \frac{n_y^2}{n_x^2} = \cot^2(\omega).$$

The smallest positive real number $k = k_\omega$ satisfying this equation is

$$\begin{aligned} k_\omega &= 1 \quad \text{if } \omega \leqslant \tfrac{\pi}{2}, \\ k_\omega &= \frac{\pi}{\omega} - 1 \quad \text{if } \tfrac{\pi}{2} < \omega < \tfrac{2\pi}{3}. \end{aligned}$$

Let $(a,b)$ a nontrivial solution of (3.18) - (3.19) with $k = k_\omega$. Then $\mathbf{w}$ is a singular function in the sense that it satisfies equations (3.9) - (3.10) with $\mathbf{g} = \mathbf{0}$. Multiplying this $\mathbf{w}$ by a suitable $\mathscr{C}^\infty$ cut off function $\varphi$ that is 1 for $r < \frac{1}{4}$ leads to a function $\bar{\mathbf{w}}$ that satisfies the boundary conditions (3.9) and is zero for $r > \frac{1}{2}$.

If $k_\omega = 1$, then $\bar{\mathbf{w}} \in H^2(\Omega_1)$, where

$$\Omega_1 = \left\{ r(\cos(\varphi), \sin(\varphi)) \mid 0 < r < 1, 0 < \varphi < \omega \right\}.$$

Otherwise, $\bar{\mathbf{w}}$ is a singularity contained in the space $H^{1+k_\omega-\varepsilon}(\Omega_1)$ for any $\varepsilon > 0$ (note that $1 + k_\omega - \varepsilon > \frac{3}{2}$ for small enough $\varepsilon$). Thus, Lemma 3.1 applies to establish $H^\alpha$ regularity for *Case 3*.

□

## 4. CONVERGENCE IN THE $H^1$ AND $L^2$ NORMS

Let $\mathbf{W}_h \subset \mathbf{W}$ and $V_h \subset V$ be finite element spaces and assume that, for every $\varepsilon \in [0,1]$, there exists a constant $C_{int}$ (independent of $h$ as well as $\mathbf{w}$ and $r$) such that

$$\inf_{\mathbf{v}_h \in \mathbf{W}_h} \|\mathbf{w} - \mathbf{v}_h\|_{1,\Omega,2} \leqslant C_{int} h^\varepsilon \|\mathbf{w}\|_{1+\varepsilon,\Omega,2} \quad \forall \mathbf{w} \in H^{1+\varepsilon}(\Omega)^2, \tag{4.1}$$

$$\inf_{q_h \in V_h} \|r - q_h\|_{1,\Omega} \leqslant C_{int} h^\varepsilon \|r\|_{1+\varepsilon,\Omega} \quad \forall r \in H^{1+\varepsilon}(\Omega). \tag{4.2}$$

**Lemma 4.1 Aubin and Nitsche.** *Assume that the bilinear form $\mathscr{F}$ is $H^\alpha$ regular for some $\alpha \in [1,2]$. Let $(\mathbf{u}, p) \in \mathbf{W} \times V$ be the solution of (2.5) and $(\mathbf{u}_h, p_h) \in \mathbf{W}_h \times V_h$ the solution of the weak discrete form*

$$\mathscr{F}(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = F(\mathbf{v}_h, q_h) \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{W}_h \times V_h. \tag{4.3}$$

*Then there exists a constant $C_{AN}$ such that*

$$\|(\mathbf{u}, p) - (\mathbf{u}_h, p_h)\|_{0,\Omega,3} \leqslant C_{AN} h^{\alpha-1} \|(\mathbf{u}, p) - (\mathbf{u}_h, p_h)\|_{1,\Omega,3}.$$

**Proof.** Let $(\mathbf{w}, r) \in \mathbf{W} \times V$ be the solution of the equation

$$\mathscr{F}(\mathbf{w}, r; \mathbf{v}, q) = ((\mathbf{u} - \mathbf{u}_h), \mathbf{v})_{0,\Omega,2} + ((p - p_h), q)_{0,\Omega} \quad \forall (\mathbf{v}, q) \in \mathbf{W} \times V. \tag{4.4}$$

By $H^\alpha$ regularity of $\mathscr{F}$, we have

$$\|(\mathbf{w}, r)\|_{\alpha,\Omega,3} \leqslant C_{reg} \|((\mathbf{u} - \mathbf{u}_h), (p - p_h))\|_{0,\Omega,3}. \tag{4.5}$$

Choosing $(\mathbf{v}, q) = ((\mathbf{u} - \mathbf{u}_h), (p - p_h))$ in (4.4) yields

$$\begin{aligned} \|(\mathbf{u}, p) - (\mathbf{u}_h, p_h)\|_{0,\Omega,3}^2 &= \mathscr{F}(\mathbf{w}, r; (\mathbf{u} - \mathbf{u}_h), (p - p_h)) \\ &= \mathscr{F}((\mathbf{w} - \mathbf{w}_h), (r - r_h); (\mathbf{u} - \mathbf{u}_h), (p - p_h)) \end{aligned}$$

for every $(\mathbf{w}_h, r_h) \in \mathbf{W}_h \times V_h$. Here we used the standard observation that

$$\mathscr{F}((\mathbf{u}-\mathbf{u}_h),(p-p_h);\mathbf{w}_h,r_h) = 0 \quad \forall(\mathbf{w}_h,r_h) \in \mathbf{W}_h \times V_h.$$

Combining (2.4) and (4.1) - (4.2) with $\varepsilon = \alpha - 1$, then using (4.5), we thus have that

$$\begin{aligned}
&\|(\mathbf{u},p)-(\mathbf{u}_h,p_h)\|^2_{0,\Omega,3} \\
&\quad \leqslant h^{\alpha-1} C_{int} \beta \|(\mathbf{w},r)\|_{\alpha,\Omega,3} \|((\mathbf{u}-\mathbf{u}_h),(p-p_h))\|_{1,\Omega,3} \\
&\quad \leqslant h^{\alpha-1} \beta C_{int} C_{reg} \|((\mathbf{u}-\mathbf{u}_h),(p-p_h))\|_{0,\Omega,3} \|((\mathbf{u}-\mathbf{u}_h),(p-p_h))\|_{1,\Omega,3}.
\end{aligned}$$

Dividing by $\|((\mathbf{u}-\mathbf{u}_h),(p-p_h))\|_{0,\Omega,3}$ completes the proof. □

**Theorem 4.1.** *Assume that the bilinear form $\mathscr{F}$ is $H^\alpha$ regular for some $\alpha \in [1,2]$. Suppose also that $p \in H^{2+\varepsilon}(\Omega)$ and $\mathbf{u} \in (H^{1+\varepsilon}(\Omega))^2$ for some $\varepsilon \in [0,1]$. Let $(\mathbf{u}_h, p_h) \in \mathbf{W}_h \times V_h$ be the solution of the weak discrete equation (4.3). Then $p_h$ converges to $p$ and $\mathbf{u}_h$ converges to $\mathbf{u}$ according to the estimates*

$$\|p-p_h\|_{1,\Omega} = O(h^{\min(\alpha-1+\varepsilon,1)}), \tag{4.6}$$

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega,2} = O(h^\varepsilon), \tag{4.7}$$

$$\|(\mathbf{u},p)-(\mathbf{u}_h,p_h)\|_{0,\Omega,3} = O(h^{\alpha-1+\varepsilon}). \tag{4.8}$$

**Proof.** Assumptions $p \in H^{2+\varepsilon}$ and $\mathbf{u} \in (H^{1+\varepsilon}(\Omega))^2$ imply that

$$\|(\mathbf{u},p)-(\mathbf{u}_h,p_h)\|_{1,\Omega,3} = O(h^\varepsilon).$$

Thus, (4.7) is immediate and (4.8) follows from Lemma 4.1. But Lemma 2.1 implies that

$$(\nabla p, \nabla q)_{0,\Omega,2} = (\mathbf{u}, \nabla q)_{0,\Omega,2} + F(0,q) =: G(q) \qquad \forall q \in V$$

and

$$(\nabla p_h, \nabla q_h)_{0,\Omega,2} = (\mathbf{u}_h, \nabla q_h)_{0,\Omega,2} + F(0,q_h) =: G_h(q_h) \qquad \forall q_h \in V_h,$$

and (4.8) implies that

$$\sup_{q_h \in V_h} \frac{|G(q_h) - G_h(q_h)|}{\|q_h\|_{1,\Omega}} = O(h^{\alpha-1+\varepsilon}).$$

Thus, (4.6) now follows from Strang's Lemma. □

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## REFERENCES

1. S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. *Comm. Pure Appl. Math.*, 17: 35–92, 1964.
2. Z. Cai, R. Lazarov, T.A. Manteuffel, and S. F. McCormick. First-order system least squares for partial differential equations: Part I. *SIAM J. Numer. Anal.*, 31: 1785–1802, 1994.
3. Z. Cai, T.A. Manteuffel, and S. F. McCormick. First-order system least squares for partial differential equations: Part II. *SIAM J. Numer. Anal.*, 34: 425–454, 1997.
4. M. Dauge. Elliptic Boundary Value Problems on Corner Domains. Lecture Notes in Mathematics 1341. Springer, 1988.
5. P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Monographs and Studies in Mathematics 24. Pitman, Boston, London, Melbourne, 1985.
6. V.A.Kondrat'ev. Boundary Problems for Elliptic Equations in Domains with Conical or Angular Points. Trans. Mosc. Math. Soc., 16:227-313, 1967.
7. V. G. Maz'ya and B. A. Plamenevskii. Coefficients in the asymptotics of the solutions of elliptic boundary-value problems in a cone. *J. Sov. Math.*, 9: 750–764, 1978.
8. V. G. Maz'ya, S. A. Nasarow, and B. A. Plamenevski. *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten I.* Akademie-Verlag, Berlin, 1991.
9. M. Renardy and R. C. Rogers. *An Introduction to Partial Differential Equations*. Springer, Berlin, Heidelberg, New York, Tokyo, 1993.
10. J. Wloka. *Partielle Differentialgleichungen*. Teubner, Stuttgart, 1982.