3. James Stewart, *Calculus* (5th ed.), Brooks/Cole, 2003.
4. *TI-83 Graphing Calculator Guidebook*, Texas Instruments,1995.

Musical Analysis and Synthesis in Matlab

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This article presents musical examples and computer tools that could be used in an introductory pde and Fourier Series class, where students work with the heat and wave equations. The study of Fourier series from a musical perspective offers great insight into basic mathematical concepts and the physics of musical instruments. Tools available in Matlab allow students to easily analyze the wave forms and harmonics of recorded sounds and to synthesize their own. These experiments are a thought provoking way to understand how a wave form is composed of a summation of Fourier Series basis functions and how this relates to the frequency domain. Many students I have worked with have special musical interests and go on to conduct experiments with their own instruments.

The plucked string. One of the standard problems in an introductory pde course is the wave equation with Dirichlet boundary conditions,

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in (0, L), t \geq 0 \\ u(0, t) = u(L, t) = 0, & t \geq 0 \\ u(x, 0) = \alpha(x), \ u_t(x, 0) = \beta(x), & x \in (0, L). \end{cases} \tag{1}$$

Physically, we can think of $u(x, t)$ as the displacement of a plucked guitar string with initial displacement $\alpha(x)$ and initial velocity $\beta(x)$. Additional terms may be added to the wave equation to account for string stiffness and internal damping [**1**]. The solution, obtained by separation of variables, is

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left(a_n \cos \frac{n\pi}{L} ct + b_n \sin \frac{n\pi}{L} ct \right), \tag{2}$$

where

$$a_n = \frac{2}{\pi} \int_0^L \alpha(x) \ \sin \frac{n\pi}{L} x \, dx \tag{3}$$

$$b_n = \frac{2L}{cn\pi^2} \int_0^L \beta(x) \ \sin \frac{n\pi}{L} x \, dx \tag{4}$$

are the Fourier coefficients of $\alpha(x)$ and $\beta(x)$. Using the trig identity $\cos(x - y) = \cos x \ \cos y + \sin x \ \sin y$ we may rewrite the solution as

$$u(x, t) = \sum_{n=1}^{\infty} p_n \ \sin \frac{n\pi}{L} x \ \cos \frac{n\pi}{L} c \, (t - \gamma_n). \tag{5}$$

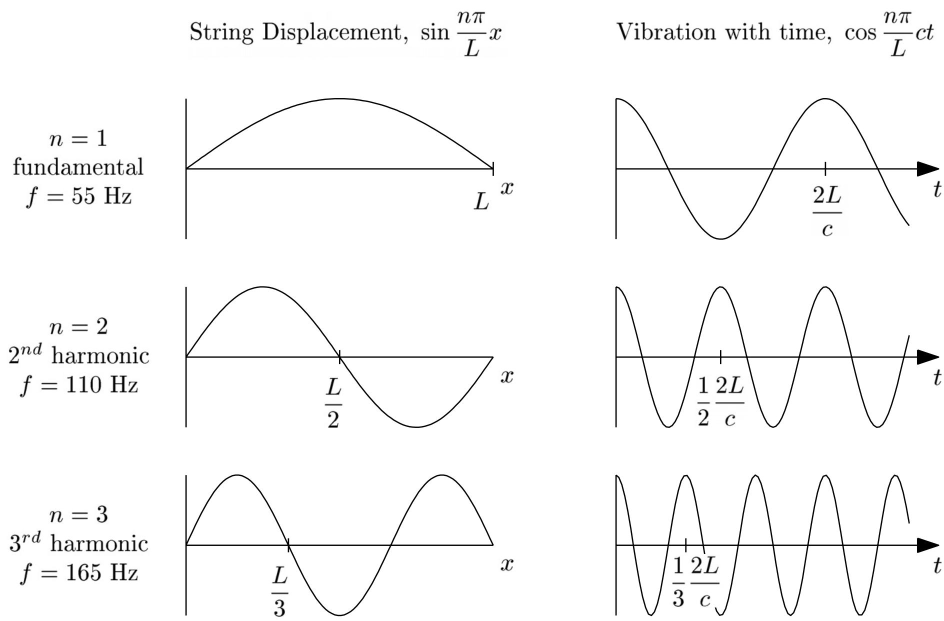

Figure 1. Modes of a vibrating string.

The solution is a superposition of many modes, where each mode oscillates at a different frequency as shown in Figure 1. The first factor in the sum, $\sin \frac{n\pi}{L} x$, is a snapshot of the vibrating string. The second factor, $\cos \frac{n\pi}{L} c(t - \gamma_n)$, shows how this mode vibrates with time. The nth mode has a frequency of $f_n = n\frac{c}{2L}$, a phase shift of γ_n, and the amplitude of each mode is $p_n = \sqrt{a_n^2 + b_n^2}$.

A musician would call f_1 the fundamental frequency and $f_2, f_3, \ldots$ its harmonics. The pitch of each note on the keyboard is associated with a specific fundamental frequency. For example, a low low low A has a fundamental frequency of 55 Hz, and harmonics occur at $110, 165, 220, 275, \ldots$. Note that the harmonics form an arithmetic sequence, $f_n = nf_1$. This is because a string with fixed endpoints can physically only vibrate in the modes shown in column 1 of Figure 1.

The wave speed c can be shown to be the square root of the string's tension over its line density [**1**, p. 36]. Thus the pitch of a string is described by three parameters,

$$f_1 = \frac{1}{2\text{ length}}\sqrt{\frac{\text{tension}}{\text{line density}}}. \tag{6}$$

This makes sense, as tightening a guitar string increases its pitch while choosing longer strings and thicker strings lowers the pitch. A guitarist changes pitch while playing by shortening the string against the frets.

The solution (5) describes a vibrating string, but other instruments like woodwinds and brass are similar. When an instrument vibrates it sets the surrounding air in motion, producing oscillations in pressure. These are the sound waves detected by your ear. Thus a microphone near our vibrating string would record

$$y(t) = \sum_{n=1}^{\infty} p_n \cos 2\pi f_n (t - \gamma_n). \tag{7}$$

Physically the amplitudes p_n describe the energy associated with each harmonic. The power spectrum is the distribution of energy among the harmonics, in this case the vector $[p_1, p_2, p_3, \ldots]$. For real sounds the power is concentrated in the first 10 to 20 harmonics and the power spectrum is a continuous function of frequency.

Harmonic analysis. At this point there is a wealth of experiments that a student (or instructor) could conduct. Questions about how the spectrum of harmonics changes for different instruments and notes can be investigated using Matlab. The function `analyze.m` [**4**] uses the built-in Matlab functions `wavread` and `fft` to calculate the power spectrum of a Microsoft wave (.wav) sound file. A similar function named `auread` can be used for UNIX audio files.

The plots produced by `analyze.m` can be used to identify the pitch and volume of a sound sample. Figure 2 shows the results of low A, middle A, and high A played on a piano. The waveforms on the left are the pressure variations with time detected by the microphone. The amplitude of the wave is a measure of its pressure oscillations and corresponds to the volume of the sound. Volume is usually measured in decibels, the logarithm of pressure. One does not usually think of sound in terms of pressure, but feeling the pressure waves of sound at a concert or near loud equipment is a common experience.

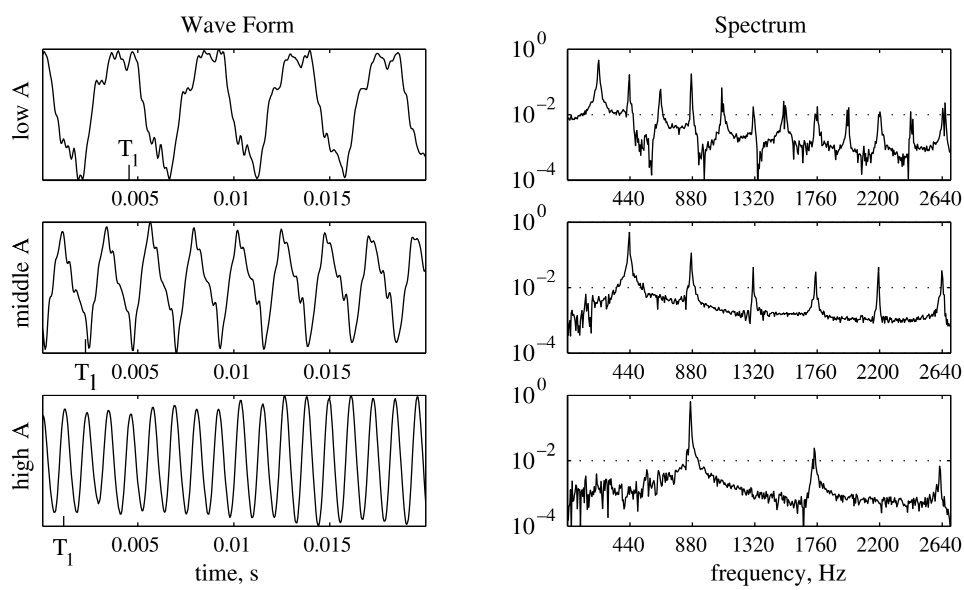

Figure 2. Analysis of piano notes using `analyze.m`

From the wave forms in Figure 2 one can immediately see the periodic nature of each sound and pick out its fundamental period T_1. The corresponding fundamental frequencies ($f_1 = 1/T_1$) are 220 Hz, 440 Hz, and 880 Hz for the three notes and can be seen as the first peaks on the power spectrums. The fundamental frequency f_1 doubles with each octave, and the harmonics are spaced in proportion to f_1 as expected. The spectrum is nonzero between harmonics because the waves are not perfectly periodic.

Notice that the difference in frequency between any two consecutive harmonics is the fundamental frequency. The human mind unconsciously uses this fact to identify a pitch even if the fundamental and lower harmonics are missing. This is how small speakers make the low sounds of a bass guitar without actually producing low frequencies (Exercise 2).

Going up an octave always corresponds to a doubling in frequency. Thus the frequencies of octaves form a geometric sequence. For example, the frequencies of As are 55, 110, 220, 440, 880 .... According to (6) an octave's doubling in frequency can be accomplished by halving the length of the string, as can be verified on any stringed instrument.

The fact that octaves form a geometric sequence has many physical manifestations, such as:

- Low instruments must be much larger than high instruments. In general, an instrument which is an octave lower must be twice as large. For example, in the string family, as we progress from violin, viola, cello, to bass, the cello is large and the bass is very large.
- Organ pipes must also double in size to go down an octave. This is why the organ pipes at the front of a church, if arranged in descending order, approximate an exponential curve.
- Frets on a guitar are far apart at the neck and close together near the body, a pattern which also appears on log graphing paper. Frets and log paper both follow an inverse exponential pattern.

The human mind can identify a seemingly infinite number of instruments by their sound alone, even if they are playing the same pitch at the same volume. What is it that differentiates the pressure signal of a flute from that of a violin? Figure 3 shows the wave form and spectrum of several instruments playing middle A (440 Hz). The wave forms look completely different, but all have the same fundamental period of 0.0023 seconds. The timbre, or sound quality of an instrument, is due to the relative strengths of the harmonics. A pure tone, composed of the fundamental alone, is shrill and metallic, like a tuning fork. Power in the higher harmonics add warmth and color to the tone. Figure 3 shows that a flute is dominated by the lower harmonics, accounting for its simple whistle-like timbre. In contrast violins have power in the higher harmonics,

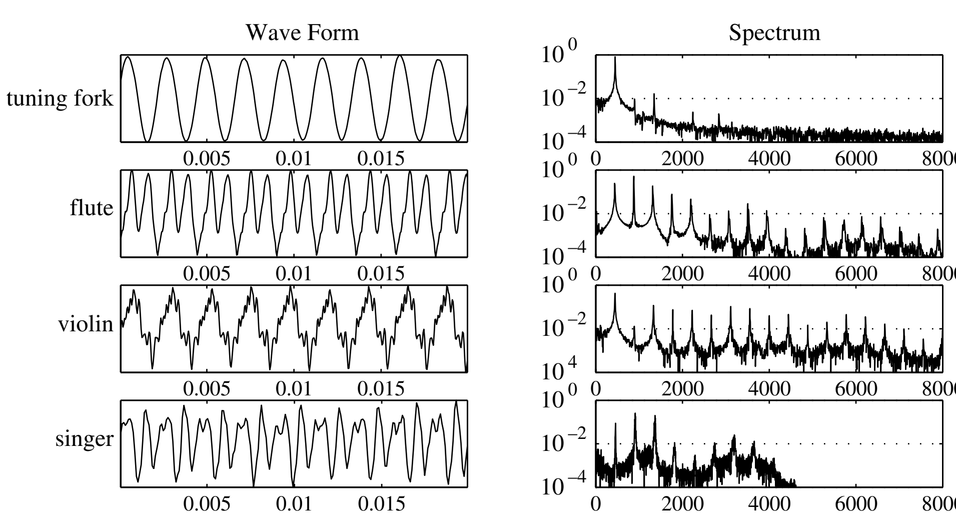

Figure 3. Analysis of several instruments using `analyze.m`

which gives them a warmer, more complex sound. The effect of higher harmonics is clearly seen in the waveform, where the violin has complex oscillations within each period.

Now consider a singer who sings different vowels at the same pitch and volume. Again, the difference between vowels is the distribution of power among the harmonics. In 1859 Helmholtz was able to detect individual harmonics using membranes and bulbs which vibrate sympathetically at specific frequencies [**2**, p. 42]. He then created a series of tuning forks tuned to the harmonics of $b^\flat$. Each fork was forced to vibrate electromagnetically while its volume was controlled by a cover. He astonished audiences by creating synthesized vowel sounds by simply adjusting the covers of these tuning forks (see Exercise 3).

Synthesis. At this point, I present the following role-play to my students:

> Imagine you are an engineer for Yamaha in 1958, and your boss comes in the office one day and says, "Can we make a keyboard that sounds like any instrument you choose? That could revolutionize the music industry!" And you say, "Yes, I know how to do that!"

We know that musical sounds can be analyzed to measure the distribution of power in the harmonics. For a particular instrument, this distribution is the key to synthesizing its sound. Given a fundamental frequency f_1 and power p_n associated with the nth harmonic, the synthesized waveform is

$$y(t) = \sum_{n=1}^{N} p_n \cos 2\pi n f_1 t. \tag{8}$$

The function `synthesize.m` [**4**] creates sound wave data in this way, and then converts the data into a wave sound file with the Matlab function `wavwrite`. Students can use this simple program to experiment with different combinations of harmonics. Higher frequencies sound louder to the human ear than lower frequencies of the same decibel level (amplitude). Specifically, when choosing the power spectrum to create a sound, frequencies between 3000 and 5000 Hz should have a tenth the amplitude of lower frequencies in order to produce the same apparent volume [**3**, p. 233].

Early synthesized music used a power spectrum which was constant in time, which gave it a false, drone sound. In reality the power spectrum changes as the musician varies volume, vibrato, and phrasing. Modern keyboards vary the power spectrum with time to account for the attack and decay of a hit piano string or plucked banjo. They also vary the spectrum for high, mid, and low ranges as real instruments do. Synthesized music has improved dramatically since the sixties, but it could never be as variable or expressive as a musician on a live instrument.

Exercises. See [**4**] for Matlab code, sample music files, and selected solutions.

1. Analyze .wav files of two different instruments. Identify the fundamental frequency in the wave form and power spectrum for each case. Do the relative amplitudes of the harmonics explain the timbres of the instruments?
2. The human mind can identify a pitch even if the fundamental and lower harmonics are missing. Synthesize sounds where $p_1 = 0$, where $p_1 = p_2 = 0$, and where $p_1 = p_2 = p_3 = 0$. What pitch do you hear? Can you still recognize the fundamental in the wave form or the power spectrum?

3. Helmholtz suggested the following power spectrums to synthesize vowels [**2**, p. 123, 543].

		p_1	p_2	p_3	p_4	p_5	p_6	p_8	p_{16}
U	oo as in boot	ff	mf	pp					
O	oh as in no	mf	f	mf	p				
A	ah as in caught	p	p	p	mf	mf	p	p	
E	eh as in bed	mf		mf			ff		
I	ee as in see	mf	p				p		mf

Use `synthesize.m` to produce these vowels.

4. According to equation 7, each mode may have a phase shift γ_n. Revise `synthesize.m` to include phase shifts, and create a sound where the second harmonic is shifted from the first. Does this shift affect the waveform, spectrum, or sound of the note? After doing this same test with electromagnetically forced tuning forks, Helmholtz concluded that phase shifts do not affect the sound.
5. When two notes with fundamental frequencies f and $\tilde{f}$ are played together and exactly on pitch, one may hear a difference tone $|f - \tilde{f}|$ which is lower than the original notes or a summation tone $f + \tilde{f}$ which is higher. Sketch the waveforms of the fundamental of a note and its fifth to explain why difference and summation tones can be heard, and try to produce them with Matlab.
6. Beats can be produced by playing two strings of slightly different frequencies simultaneously. Plot $\sin 2\pi f t$, $\sin 2\pi(f + \epsilon)t$, and their sum to show this effect. Use a trig identity to rewrite the sum as the product of a fast wave and a slow wave.
7. Helmholtz said 33 beats per second is the most painful beat rate to listen to. If $f = 261.63$ Hz (middle C) what lower frequency, when played with f, produces 33 beats per second? Do the same for $f = 523.25$ Hz (high C) and see what notes your calculated frequencies correspond to. Create these beats with Matlab and see if you agree with Helmholtz.

References

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2. H. Helmholtz, *On the Sensations of Tone* (4th ed.), Dover, 1980. Translated by A. J. Ellis, originally published 1877.
3. I. Johnston, *Measured Tones: The Interplay of Physics and Music* (2nd ed.), Institute of Physics Publishing, 2002.
4. Mark R. Petersen, code and music samples may be found at http://amath.colorado.edu/pub/matlab/music