First-Order System Least Squares FOSLS

DOE & NSF & IBM

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HISTORICAL GAP

fast solvers

&

uniform performance

Reynolds number, Lamé constants,...

Outline

no frills

I. Basic FOSLS

II. FOSLS Tools

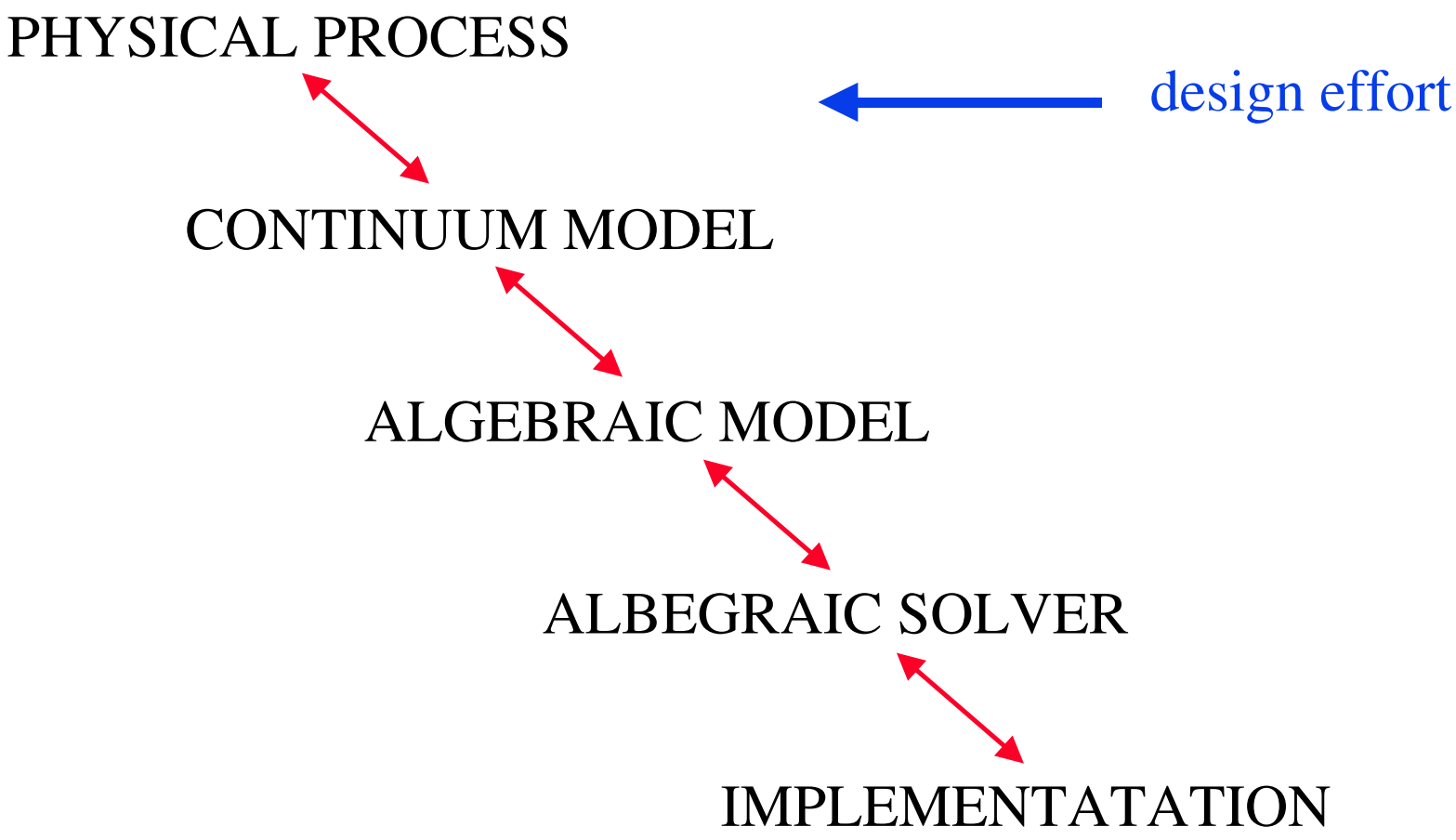
III. Anatomy of FOSLS for Convection Diffusion

IV. FOSLL*

V. Concluding Remarks

I. Basics
II. Tools
III. C-D
IV. LL*
V. End

FOSLoSophy



I. Basics
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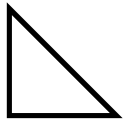
Nice Matrices: SPD + ELLIPTIC

symm. pos. def. Poisson-like

$Lq = \mathrm{f}$

SPD is not necessarily nice: $L = (-\Delta)^2$

Non-SPD is not necessarily naughty: $L =$ 

SPD + "Elliptic" = Nice:

♥ elliptic energy minimization principle ♥

$F(q) \approx \langle q, q \rangle - 2\langle q, f \rangle$

$\Rightarrow$ accurate/stable **FE** & fast solvers like **MG**

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Diffusion Example

$$Lp \equiv -p'' = f$$

dirichlet boundary

Energy:

$$F(p;f) \equiv \text{“}<Lp,p>\text{”} - 2<f,p>$$

$$= \int \left((p')^2 - 2fp \right) dx$$

weak form

Optimization: $H^1 \equiv \{p : p, p' \in L^2\}$

$$p \in H^1 \ni F(p;f) = \min_{q \in H^1} F(q;f)$$

Equivalence:

$$\nabla F(p;f) = 2(Lp - f) \;\&\; F''(p;f) >> 0$$

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Properties

Weak Form : on H^1

$$\int (p'\, q' - fq)dx = 0$$

$$\text{“}\langle Lp - f, q\rangle\text{”} = 0$$

Error Measure : p = exact minimizer

$$E(e) \equiv F(p+e;f) - F(p;f)$$

$$= \int (e')^2 dx = \|e'\|_0^2 \equiv |e|_1^2$$

Ellipticity : $E(e) \sim \|e\|_1^2 \equiv \|e\|_0^2 + \|e'\|_0^2$

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Discretization

Rayleigh – Ritz: on $S^h \subset H^1$

$$F(p^h; f) = \min_{q^h \in S^h} F(q^h; f)$$

Error (Cea's Lemma):

$$\left\| p - p^h \right\|_1 \leq C \min_{q^h \in S^h} \left\| p - q^h \right\|_1 \leq Ch \left\| p \right\|_2$$

Nitsche's Trick:

$$\left\| p - p^h \right\|_0 \leq Ch^2 \left\| p \right\|_2$$

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Multigrid

$$\min F(q^h)$$

Relaxation: $\varepsilon_i^h \in S^h \subset H^1$

$$F(q^h + s\varepsilon_i^h; f) = \min_{t \in \mathbf{R}} F(q^h + t\varepsilon_i^h; f)$$

Coarsening: $S^{2h} \subset S^h$

$$F(q^h + p^{2h}; f) = \min_{q^{2h} \in S^{2h}} F(q^h + q^{2h}; f)$$

Optimality (Oswald, Zhang, Bramble, ...):

homogeneous/bilinear part

$$F(p+e; f) - F(p; f) = E(e) \sim \|e\|_1^2$$

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Minimization Principle Advantages

$\min F(q)$

- sense of optimality
- error monitor $F(q)$? $\min F(q)$?
- monotone: $T \supset S \Rightarrow \min_T F \leq \min_S F$
- continuum functions & subspaces
- approximation theory analysis
- elliptic $\Rightarrow$
  - standard FE & MG with theory
  - final H^1 accuracy

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FOSLS Essence

Motivation: purely elliptic problems are well understood

♥ scalar elliptic energy minimization principle ♥

Plan: imbue PDE with energy principle

Objective: PDE → loosely coupled elliptic equations

Purpose: *standard* finite elements and multigrid methods

Approach: PDE → First Order System → Energy

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Convection Example

$$Mp \equiv -p' = g$$

ignore boundary

Functional :

$$F(p;g) \equiv \|Mp - g\|_0^2$$

$$= \int \left((p')^2 - 2pg' + g^2 \right) dx$$

$$= \text{“}\langle Lp, p \rangle\text{”} - 2\langle p, g' \rangle + C + \text{boundary terms}$$

Normal Equation :

$$-p'' = f + \text{boundary terms}$$

I. Basics
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FOSLS for Convection-Diffusion

$$-ap'' - p' = f$$

ignore boundary

Least Squares:

$$\text{minimize} \quad \| ap'' + p' + f \|^2$$

Formal Normal:

$$(a\frac{d^2}{dx^2} - \frac{d}{dx})(a\frac{d^2}{dx^2} + \frac{d}{dx})p = (a\frac{d^2}{dx^2} - \frac{d}{dx})f$$

!!! fourth order !!!

I. Basics
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FOSLS for Convection-Diffusion

$$-ap'' - p' = f$$

ignore boundary

second order

First – Order System: $u = -ap'$

$$u + ap' = 0$$

$$u' - \quad p' = f$$

FOSLS Functional:

$$F(u, p; f) \equiv \|u + ap'\|^2 + \|u' - p' - f\|^2$$

Error Measure: $E(e_u, e_p)$ homogeneous part

$$F(u + e_u, p + e_p; f) - F(u, p; f) \approx \|e_u\|_1^2 + \|e_p\|_1^2$$

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Product H^1 Equivalence

$$F = \|\mathcal{L}w - f\|^2 = \|\mathcal{L}(w - w^*)\|^2 = \|\mathcal{L}e\|^2$$

$$\Rightarrow \quad F(u + e_u, p + e_p; f) = F(e_u, e_p; 0)$$

study $f = 0$ case

$$F(u, p; 0) = \|u + ap'\|^2 + \|u' - p'\|^2$$

$$\Rightarrow \quad F(u, p; 0) \sim \|u\|^2 + \|u'\|^2 + \|p'\|^2$$

$$\Rightarrow \quad F(u, p; 0) \sim \|u\|_1^2 + |p|_1^2$$

$$\Rightarrow \quad \min \; F(u, p; f) \;\Leftrightarrow\; 2 \text{ Poisson}$$

$$\Rightarrow \quad \text{standard FE} \quad \& \quad \text{MG}$$

I. Basics
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Error Measure

Objective: approximation + *reasonable* error measure

Ideal: optimal approximation in the measure

Reasonable:

- always accurate values: L^2
- often accurate derivatives: H^1

Conventional Measure:

- residual = $\|Lp\text{-}f\|_0 = \|L(p\text{-}p^*)\|_0 = \|Le\|_0 = \|e'\|_0$
- oscillatory error $\rightarrow$ overly large residuals

I. Basics
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FOSLS Error Measure

for *any* approximation **w**
to the exact solution **w***

Absolute :

$$F(\mathbf{w};\mathbf{f}) = \|L\mathbf{w}-\mathbf{f}\|_0^2 = \|L(\mathbf{w}-\mathbf{w}^*)\|_0^2$$

$$\sim \|\mathbf{w}-\mathbf{w}^*\|_1^2$$

Relative :

$$\frac{F(\mathbf{w};\mathbf{f})}{F(\mathbf{0};\mathbf{f})} = \frac{\|L\mathbf{w}-\mathbf{f}\|_0^2}{\|L\mathbf{w}^*\|_0^2} \sim \frac{\|\mathbf{w}-\mathbf{w}^*\|_1^2}{\|\mathbf{w}^*\|_1^2}$$

!!! powerful tool for adaptive refinement !!!

!avoids patch test for nonconforming elements!

I. Basics
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#1: New Variables

reducing order

Poisson:

$$\nabla \bullet \nabla p = f$$

$$\Rightarrow \mathbf{u} = \nabla p$$

CFD/Elasticity:

$$-\Delta \mathbf{u} + \mathrm{Re}(\nabla \mathbf{u}^t)^t \mathbf{u} + \nabla p = f$$

$$\nabla \bullet \mathbf{u} + \alpha p = g$$

$$\Rightarrow \mathbf{U} = \nabla \mathbf{u}^t$$

I. Basics
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#2: New Equations

good norm equivalence

Poisson :

$$\mathbf{u} - \nabla p = \mathbf{0}$$

$$\nabla \bullet \mathbf{u} + f = 0$$

$$\Rightarrow \quad \nabla \times \mathbf{u} = \mathbf{0}$$

CFD/Elasticity :

$$\mathbf{U} - \nabla \mathbf{u}^t = \mathbf{O}$$

$$-\nabla \bullet \mathbf{U} + \mathrm{Re}\, \mathbf{U}^t \mathbf{u} + \nabla p = \mathbf{f}$$

nonlinearity across variables only

$$\nabla \bullet \mathbf{u} + \alpha p = g$$

$$\Rightarrow \quad \nabla \times \mathbf{U} = \mathbf{O}$$

$$\& \quad \nabla \mathit{trace}\, \mathbf{U} + \alpha \nabla p = \mathbf{0}$$

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Curl Option

Poisson system :

$$\mathbf{u} - \nabla p = \mathbf{0}$$
$$\nabla \bullet \mathbf{u} = f$$

$$\nabla \times \mathbf{u} = \mathbf{0}$$

Div - free error : $\nabla \bullet \mathbf{e} = 0$

$$\mathbf{u} + \mathbf{e} - \nabla p - \mathbf{0} = \mathbf{e}$$
$$\nabla \bullet (\mathbf{u} + \mathbf{e}) - f = 0$$

$$\nabla \times (\mathbf{u} + \mathbf{e}) = \nabla \times \mathbf{e}$$

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#3: New Boundary Conditions

good norms

Poisson Case : $\mathbf{u} = \nabla p$

Neumann :

$$\mathbf{n} \bullet \nabla p = 0 \quad \Rightarrow \quad \mathbf{n} \bullet \mathbf{u} = 0$$

Dirichlet :

$$p = 0 \quad \Rightarrow \quad \mathbf{n} \times \mathbf{u} = \mathbf{0}$$

Robin/Radiation :

$$\mathbf{n} \bullet \nabla p + \alpha p = 0 \quad \Rightarrow \quad \mathbf{n} \bullet \mathbf{u} + \alpha p = 0$$

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Tangential Boundary Condition

$$p = 0$$

$$p_x = 0$$

$$\nabla p \parallel \mathbf{n}$$

$$\mathbf{n} \times \nabla p = 0$$

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#4: Preconditioners

nonsmooth data or L^2 measure

ILLUSTRATE WITH POISSON:

$$\mathbf{u} - \nabla p = \mathbf{0}$$

$$(-\Delta)^{-\frac{1}{2}}(\nabla \bullet \mathbf{u} + f) = 0$$

$$(-\Delta)^{-\frac{1}{2}}(\nabla \times \mathbf{u}) = \mathbf{0}$$

$$F_{-1}(\mathbf{u}, p; f) =$$

$$\|\mathbf{u} - \nabla p\|_0^2 + \|\nabla \bullet \mathbf{u} - f\|_{-1}^2 + \|\nabla \times \mathbf{u}\|_{-1}^2$$

$$\Rightarrow \qquad F_{-1}(\mathbf{u}, p; 0) \sim \|\nabla p\|_0^2 + \|\mathbf{u}\|_0^2$$

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Inverse Norm

Precondition $L\mathbf{w} = \mathrm{f}$ by $B = (-\Delta)^{-1/2}$:

$$BL\mathbf{w} = B\,\mathbf{f}$$

$$\|q\|_{-1}^2 \equiv \sup_{r \in H^1} \frac{< q, r >^2}{\|r\|_1^2}$$

$$= < (-\Delta)^{-1} q, q >$$

$$= \left\| (-\Delta)^{-1/2} q \right\|_0^2$$

direct computation by MG

I. Basics
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#5: Scaling

balanced equivalence

Poisson Case: $\mathbf{u} = p$

$$F_\varepsilon(\mathbf{u}, p; f) =$$

$$\varepsilon\|\mathbf{u} - p\|^2 + (1-\varepsilon)(\| \mathbf{u} + f\|^2 + \| \infty\mathbf{u}\|^2)$$

Stage I:

$$F_0(\mathbf{u}; f) = \| \mathbf{u} + f\|^2 + \| \infty\mathbf{u}\|^2$$

$$F_0(\mathbf{u}; 0) \sim \| \mathbf{u}\|^2$$

StageII:

$$F_1(p; \mathbf{u}) = \| p - \mathbf{u}\|^2$$

$$F_1(p; \mathbf{0}) = \| p\|^2$$

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FOSLS for Scalar PDEs

convection diffusion

$$-\nabla \bullet \nabla p + a p_x = f, \qquad 0 \le x, y \le 1$$

$$p(0,y) = p(1,y) = 0$$

$$p_y(x,0) = p_y(x,1) = 0$$

Guide

$$-\nabla \bullet \nabla p + a p_x = -e^{ax}\left(e^{-ax} p_x\right)_x - p_{yy}$$

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Convection Diffusion 1-3

#1: New Variables

$$u = e^{-ax} p_x$$

flux

$$v = \quad p_y$$

#2: New Equations

$$-e^{ax} u_x - v_y + f = 0$$ div

$$e^{ax} u_y - v_x = 0$$ 'curl'

#3: New Boundary Conditions

$$p(0, y) = p(1, y) = 0$$

$$v(0, y) = v(1, y) = 0$$

$$v(x, 0) = v(x, 1) = 0$$

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Convection Diffusion 5

tool #5

$f = 0$ for simplicity

First Stage :

$$- e^{ax} u_x - v_y = 0$$

$$e^{ax} u_y - v_x = 0$$

Troublesome Scaling :

$e^{-ax/2}$

$$\left\| -e^{ax} u_x - v_y \right\|^2 + \left\| e^{ax} u_y - v_x \right\|^2$$

$$= \left\| e^{ax} u_x \right\|^2 + 2 < e^{ax} u_x, v_y > + \left\| v_y \right\|^2$$

$$+ \left\| e^{ax} u_y \right\|^2 - 2 < e^{ax} u_y, v_x > + \left\| v_x \right\|^2$$

decoupled *u* & *v* ?

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Convection Diffusion 5

continued

Better Scaling : $s_{\pm}(x,y) = e^{\pm ax/2}$

$$\left\| -s_+ u_x - s_- v_y \right\|^2 + \left\| s_+ u_y - s_- v_x \right\|^2$$

$$= \left\| s_+ u_x \right\|^2 + 2 < u_x, v_y > + \left\| s_- v_y \right\|^2$$

$$+ \left\| s_+ u_y \right\|^2 - 2 < u_y, v_x > + \left\| s_- v_x \right\|^2$$

- ***uniform*** scaled H^1 equivalence
- scaling damps boundary layer behavior
- valid for more general convection term
- Poisson equation for recovering p

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More General Form

$$s_- \sim e^{-ax/2}$$

$$\mathbf{u} = \nabla p$$

$$-\nabla \bullet \mathbf{u} + au = f$$

$$F_0(\mathbf{u}; f) = \| s_- [-\nabla \bullet \mathbf{u} + au - f] \|^2 + \| s_- [\nabla \times \mathbf{u}] \|^2$$

$$F_0(\mathbf{u}; 0) \sim \| s_- \nabla \mathbf{u} \|^2 + \| s_- \nabla \mathbf{v} \|^2$$

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IV. FOSLL*

FOSLS Symmetrization: L^*L

$$Lu = f$$

Residual:

$$Lu^* = f \;\Rightarrow\; Lu - f \;=\; L(u - u^*) \;=\; Le$$

Functional:

$$\|Lu - f\|^2 \;=\; \|Le\|^2 \;=\; \langle Le, Le\rangle \;=\; \langle L^*Le, e\rangle$$

Formal Normal:

$$\nabla \|Lu - f\|^2 \;=\; 2L^*(Lu - f)$$

Symmetrization

basic FOS

FOSLS: L^*L

$$\begin{pmatrix} I & -\nabla \\ \nabla\bullet & 0 \end{pmatrix} \rightarrow \begin{pmatrix} I & -\nabla \\ \nabla\bullet & 0 \end{pmatrix}^* \cdot \begin{pmatrix} I & -\nabla \\ \nabla\bullet & 0 \end{pmatrix}$$

FOSLL*: LL^*

$$\begin{pmatrix} I & -\nabla \\ \nabla\bullet & 0 \end{pmatrix} \rightarrow \begin{pmatrix} I & -\nabla \\ \nabla\bullet & 0 \end{pmatrix} \cdot \begin{pmatrix} I & -\nabla \\ \nabla\bullet & 0 \end{pmatrix}^*$$

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Augmenting FOS for Ellipticity

FOSLS: L^*L

$$\begin{pmatrix} I & -\nabla \\ \nabla\bullet & 0 \\ \nabla\times & 0 \end{pmatrix} \rightarrow \begin{pmatrix} I & -\nabla \\ \nabla\bullet & \mathbf{0} \\ \nabla\times & \mathbf{0} \end{pmatrix}^* \begin{pmatrix} I & -\nabla \\ \nabla\bullet & \mathbf{0} \\ \nabla\times & \mathbf{0} \end{pmatrix} = \begin{pmatrix} I-\Delta & -\nabla \\ \nabla\bullet & -\Delta \end{pmatrix}$$

more equations

FOSLL*: LL^*

$$\begin{pmatrix} I & -\nabla & \nabla\times \\ \nabla\bullet & \mathbf{0} & \mathbf{0} \end{pmatrix} \rightarrow \begin{pmatrix} I & -\nabla & \nabla\times \\ \nabla\bullet & \mathbf{0} & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} I & -\nabla & \nabla\times \\ \nabla\bullet & \mathbf{0} & \mathbf{0} \end{pmatrix}^*$$

more variables

$$= \begin{pmatrix} I-\Delta & -\nabla \\ \nabla\bullet & -\Delta \end{pmatrix}$$

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FOSLL*

inverse norm based on L

B self-adjoint positive-definite

$$\Rightarrow \quad \|B^{-1/2}(Lu - f)\|^2 = \|B^{-1/2}Le\|^2$$

$$= < B^{-1/2}Le, B^{-1/2}Le >$$

$$= < L^*B^{-1}Le, e >$$

L & L^* invertible and $B = LL^*$

$$\Rightarrow \qquad \|B^{-1/2}(Lu - f)\|^2 = \|e\|^2$$

I. Basics
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FOSLL*

avoids $(LL^*)^{-1}$

$Lu = f$ solvable $\Rightarrow$ $\exists\, u^*$ in $\eta^{\perp}(L) = \Re(L^*)$: $u^* = L^*w^*$

$$Lu^* = f \quad \Rightarrow \quad LL^*w^* = f$$

$\Rightarrow$ minimize $F(w; f) = \langle LL^*w, w \rangle - 2\langle w, f \rangle$

$$F(w; f) = \langle LL^*w, w \rangle - 2\langle w, LL^*w^* \rangle$$

$$= \langle L^*w, L^*w \rangle - 2\langle L^*w, L^*w^* \rangle$$

$$+ \langle L^*w^*, L^*w^* \rangle - \langle L^*w^*, L^*w^* \rangle$$

$\Rightarrow$ $F(w; f) = \|L^*(w - w^*)\|^2 + C$

$= \|u - u^*\|^2 + C$

I. Basics
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FOSLS	vs	FOSLL*
L^*L	operator	LL^*
$L^*L \approx -\Delta$	aim	$LL^* \approx -\Delta$
boundary conditions for new variables	concern for FOS	boundary conditions for new variables
equations	augmenting FOS	variables
consistency	concern	solution is in $\Re(L^*)$

use dual FOSLS principles for FOSLL*

I. Basics
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FOS for $\nabla \bullet \nabla p = f$

+ boundary conditions

$$\mathbf{u} - \nabla p = \mathbf{0}$$

$$\nabla \bullet \mathbf{u} = f$$

for Dirichlet case: $p = 0$ & $\mathbf{n} \times \mathbf{u} = \mathbf{0}$

for Neumann case: $\mathbf{n} \cdot \mathbf{u} = 0$

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FOSLS

$$\mathbf{u} - \nabla p = \mathbf{0}$$

$$\nabla \bullet \mathbf{u} = f$$

$$\nabla \times \mathbf{u} = \mathbf{0}$$

$$L = \begin{pmatrix} I & -\nabla \\ \nabla \bullet & \mathbf{0} \\ \nabla \times & \mathbf{0} \end{pmatrix}$$

duality

$$\nabla \bullet \nabla p = f$$

$$L^* L = \begin{pmatrix} I - \nabla & -\nabla \\ \nabla \bullet & -\Delta \end{pmatrix} = L L^*$$

FOSLL*

$$\mathbf{u} - \nabla p + \nabla \times \mathbf{v} = \mathbf{0}$$

$$\nabla \bullet \mathbf{u} = f$$

$$L = \begin{pmatrix} I & -\nabla & \nabla \times \\ \nabla \bullet & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

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Augmenting FOS for FOSLS

consistency

$$\mathbf{u} - \nabla p = \mathbf{0}$$

$$\nabla \bullet \mathbf{u} = f$$

no problem adding curl condition:

$$\nabla \times \mathbf{u} = \nabla \times \nabla p = \mathbf{0}$$

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Augmenting FOS for FOSLL*

solution is in $\Re(L^*)$

$u = L^* w$

$$\mathbf{0} = \mathbf{u} - \quad p + \quad \infty \mathbf{v}$$

$$\begin{pmatrix} \mathbf{u} \\ p \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} I & -\nabla \\ \nabla \bullet & \mathbf{0} \\ \nabla \times & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{0} = \mathbf{w} - \quad q - \quad \mathbf{w} + \quad \infty \quad \infty \mathbf{w}$$

$$\Rightarrow \quad \mathbf{0} = \quad \infty \mathbf{w} + \quad \infty \quad \infty \quad \infty \mathbf{w}$$

$$\Rightarrow \quad \mathbf{v} = \quad \infty \mathbf{w} = \mathbf{0}$$

$$\Rightarrow \quad \mathbf{u} = \quad p$$

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Augmenting FOS for FOSLL*

boundary conditions for slack vars

$$u = L^* w$$

$$\begin{pmatrix} \mathbf{u} \\ p \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} I & -\nabla \\ \nabla\bullet & \mathbf{0} \\ \nabla\times & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix}$$

for Dirichlet case:

$$p = 0 \;\&\; \mathbf{n}\times\mathbf{u} = \mathbf{0} \;\&\; q = 0 \;\&\; \mathbf{n}\times\mathbf{w} = \mathbf{0}$$

for Neumann case:

$$\mathbf{n}\cdot\nabla p = 0 \;\&\; \mathbf{v} = \mathbf{0} \;\&\; \mathbf{n}\cdot\mathbf{u} = 0 \;\&\; \mathbf{n}\cdot\mathbf{w} = \mathbf{n}\cdot\nabla q = 0$$

$\Rightarrow$ dual vars same as FOS vars (+ **v**)

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$FOSLL_s^*$

two-stage for C-D

problem:

$$\nabla \bullet A \nabla p + \mathbf{b} \bullet \nabla p = f$$

augmented FOS:

$$\mathbf{u} - A^{1/2} \nabla p = \mathbf{0}$$

$$\nabla \bullet A^{1/2}\mathbf{u} + \mathbf{b} \bullet A^{-1/2}\mathbf{u} = f$$

$$\nabla \times A^{-1/2}\mathbf{u} - \nabla \mathbf{v} = \mathbf{0}$$

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$FOSLL_s^*$

iterative two-stage
for C-D-R

problem:

$$\nabla \cdot A \nabla p + \mathbf{b} \cdot \nabla p + cp = f$$

augmented FOS:

$$\mathbf{u} - A^{1/2} \nabla p = \mathbf{0}$$

$$\nabla \cdot A^{1/2}\mathbf{u} + \mathbf{b} \cdot A^{-1/2}\mathbf{u} = f - cp$$

$$\nabla \times A^{-1/2}\mathbf{u} - \nabla \mathbf{v} = \mathbf{0}$$

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II. Tools
III. C-D
IV. LL*
V. End

$FOSLL_s^*$

first stage

$$\begin{pmatrix} \nabla\bullet A^{1/2}+\mathbf{b}\bullet & \mathbf{0} \\ \nabla\times A^{-1/2} & -\nabla \end{pmatrix}\cdot\begin{pmatrix} -A^{1/2}\nabla+\mathbf{b} & A^{-1/2}\nabla\times \\ \mathbf{0} & \nabla\bullet \end{pmatrix}$$

$$=\begin{pmatrix} LL^* & X^* \\ X & M \end{pmatrix}$$

where

$$L^*=-A^{1/2}\nabla+\mathbf{b} \quad \& \quad X=\nabla\times A^{-1/2}\mathbf{b}$$

&

$$M=\nabla\times A^{-1}\nabla\times -\nabla\nabla\bullet \approx -\Delta$$

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Numerical Example

two stage

problem:

$$\nabla \bullet a \nabla p + \mathbf{b} \bullet \nabla p = f, \qquad a = \begin{cases} 1, & x \le 0.5 \\ \sigma, & x > 0.5 \end{cases}$$

solution:

$$p = \begin{cases} ((2\sigma - 4)x^2 + (4 - \sigma))\sin(\pi y), & x \le 0.5 \\ (-6x^2 + 7x - 1)\sin(\pi y), & x > 0.5 \end{cases}$$

method:

FE uniform linears & AMG W(1,1) cycles

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Results

fine grids: h=1/128 $\mathbf{b}^t = (0,0)$ & $\mathbf{b}^t = (6,9)$

jump σ	‖MG‖	FE accuracy
1	0.08	$O(h^2)$ for p in L^2
10	0.13	$O(h)$ for p in H^1
100	0.25	$O(h)$ for $\mathbf{u}$ in L^2

convection scales: $\mathbf{b}^t = (6,9)$

h =	1/4	1/8	1/16	1/32	1/64	1/128
‖MG‖ =	0.34	0.64	0.51	0.31	0.15	0.8

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FOSLS/FOSLL* Advantages

- minimization principle advantages
- systematic once functional is properly posed
- 'decoupled' Poisson equation in each variable
- avoids LBB/staggering/artificial diffusion/stabilization...
- final H^1 or L^2 accuracy of all variables
- F is its own practical performance measure
- generally excellent numerical performance

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FOSLS/FOSLL* Disadvantages

- possibly more variables
  - but possibly smaller grids
- H^1 equivalence for FOSLS needs smoothness/regularity
  - but FOSLL* does not
- conservation is not automatic
  - but can be done
- very tricky to get the functional & spaces just right

✓ new variables & equations + careful scaling

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FOSLS Applications

various stages of development

- diffusion-convection
- neutron transport
- Stokes & Navier-Stokes
- elasticity
- electromagnetics
- eigenproblems
- elliptic grid generation
- boundary functionals
- discontinuities
- highly convective flows
- inverse problems
- time-dependent problems
- coupled systems: flow/structure/em/egg

FURTHER INFO

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ftp://amath.colorado.edu/pub/fosls

http://amath.colorado.edu/appm/faculty/stevem

I. Basics
II. Tools
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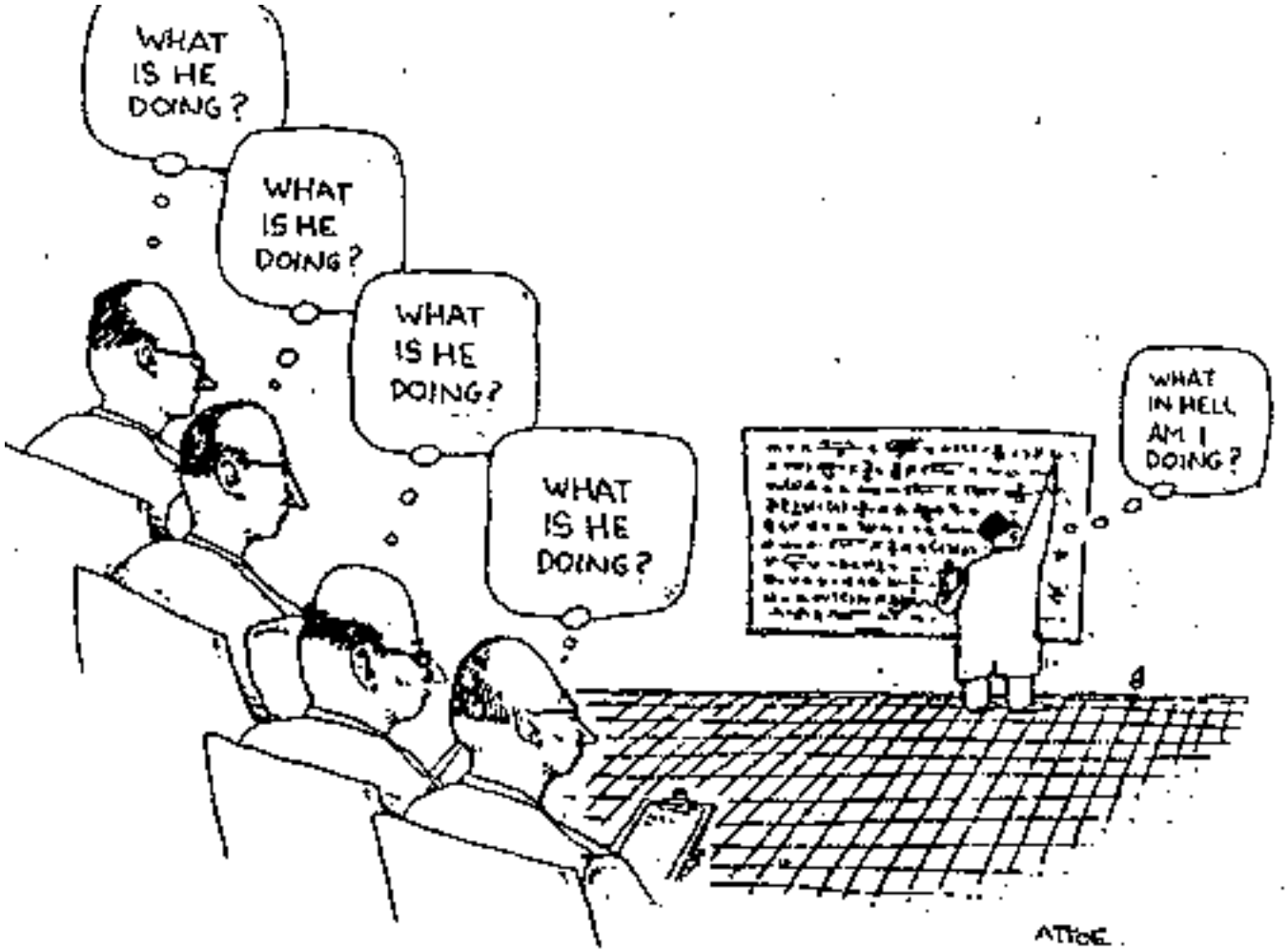
IV. LL*
V. End!

How am I doing?