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First-Order System Least Squares
FOSLS
for
Planar Linear Elasticity

by

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Outline

I. FOSLS Essence

II. Linear Elasticity

III. FOSLS Displacement Flux

IV. Norm Equivalence

V. Practical Matters

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I. FOSLS Essence

Abstract Principle

Recast the problem as a simple minimization principle so that standard approximation methods are optimal (i.e., discretization & iteration are fast & accurate).

Typical Approach

Choose new variables (to lower derivatives) & equations & scaled Sobolev norms (for balance) so the functional is almost the sum of scalar energy functionals:

$$F(p_1, p_2, \ldots, p_n) \sim \Sigma F_k(p_k)$$

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$FOSLS_1$ = Poissonification

loose collection

of scalar Poisson equations

$$-\Delta p_k = f_k$$

uniformly small coupling

\+ fast multigrid solvers

\+ accurate values & derivatives

\- smooth solutions only

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FOSLS$_0$ = Identity Equivalence

loose collection

of identities

$p_k = g_k$

- optimal preconditioners
- accurate values only
+ smooth & nonsmooth solutions

II. Planar Linear Elasticity

PURE TRACTION BOUNDARIES

$$-\Delta \mathbf{u} - (\lambda + 1)\nabla\nabla \bullet \mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega$$
$$\mathbf{n} \bullet \sigma(\mathbf{u}) = \mathbf{0} \quad \text{on} \quad \partial\Omega$$

stress tensor σ, unit normal $\mathbf{n}$, Lame' constants λ, $\mu = 1$

$$\mathbf{f} \in L^2(\Omega)^2 \cap \left\{ \begin{pmatrix} a + cy \\ b - cx \end{pmatrix} \right\}^{\perp}$$

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = (H^1(\Omega)/1)^2$$

$$\nabla \mathbf{u} = \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix} \in L^2(\Omega)^4 \cap \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}^{\perp}$$

Compact Form

$$\boxed{\begin{aligned} -\nabla \bullet A\nabla \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega \\ \mathbf{n} \bullet A\nabla \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega \end{aligned}}$$

$$A = \begin{pmatrix} \lambda+2 & 0 & 0 & \lambda \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \lambda & 0 & 0 & \lambda+2 \end{pmatrix}$$

A is singular but $\nabla \bullet A \nabla$ is nonsingular
(up to infinitesimal rotations)

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Regularity

Ω convex polygonal or $\partial\Omega$ smooth

$$\Rightarrow H^2 :$$
$$\|\nabla \bullet A\nabla \mathbf{u}\|_0 \geq \gamma \|\mathbf{u}\|_2$$

Ω bounded, open, connected & $\partial\Omega$ Lipschitz

$$\Rightarrow H^1 :$$
$$\|\nabla \bullet A\nabla \mathbf{u}\|_{-1} \geq \gamma(\|\mathbf{u}\|_1 + \lambda\|\nabla \bullet \mathbf{u}\|_0)$$

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Inverse Norms

$$H_0^1 = \{q \in H^1 : q = 0 \text{ on } \partial\Omega\}$$

Scalar norm (vector norm similar)

$$\|p\|_{-1} = \sup_{q \in H^1} \frac{< p, q >}{\|q\|_1}$$

$$\|p\|_{-1,0} = \sup_{q \in H_0^1} \frac{< p, q >}{\|q\|_1}$$

Heuristics

$$\|p\|_{-1} = \sqrt{< (I - \Delta)^{-1} p, p >}$$
$$= \quad \|(I - \Delta)^{-1/2} p\|_0$$

III. Simplest FOSLS System

DISPLACEMENT FLUX

$$\mathbf{U} = \nabla \mathbf{u} = (u_x, u_y, v_x, v_y)^t$$

$$\begin{aligned} -\nabla \bullet A\mathbf{U} &= \mathbf{f} \quad \text{in } \Omega \\ \nabla \times \mathbf{U} &= \mathbf{0} \quad \text{in } \Omega \\ \mathbf{n} \bullet A\mathbf{U} &= \mathbf{0} \quad \text{on } \partial\Omega \end{aligned}$$

FOSLS

$$\mathbf{X} = \{\mathbf{U} : \mathbf{n} \bullet A\mathbf{U} = \mathbf{0} \text{ on } \partial\Omega;\ \mathbf{U} \perp \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}\}$$

$$H^1(\Omega)^4 = \{\mathbf{U} \in L^2(\Omega)^4 : \nabla\mathbf{U} \in L^2(\Omega)^8\}$$

$$H^{divA}(\Omega) = \{\mathbf{U} \in L^2(\Omega)^4 : \nabla \bullet A\mathbf{U} \in L^2(\Omega)^2\}$$

$$F_1(\mathbf{U};\mathbf{f}) = \|\mathbf{f} + \nabla \bullet A\mathbf{U}\|_0^2 + \|\nabla \times \mathbf{U}\|_0^2$$
$$\mathbf{U} \in H^1(\Omega)^4 \cap \mathbf{X}$$

$$F_0(\mathbf{U};\mathbf{f}) = \|\mathbf{f} + \nabla \bullet A\mathbf{U}\|_{-1}^2 + \|\nabla \times \mathbf{U}\|_{-1,0}^2$$
$$\mathbf{U} \in H^{divA}(\Omega) \cap \mathbf{X}$$

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The Curl Imperative

$$F(A^\dagger \nabla \times \mathbf{V}; \mathbf{0})$$
$$= \|\nabla \bullet \nabla \times \mathbf{V}\|_0^2 + \|\nabla \times A^\dagger \nabla \times \mathbf{V}\|_0^2$$

($\nabla \bullet \nabla \times \mathbf{V} \to \mathbf{0}$)

$\Rightarrow F(\bullet; \mathbf{f})$ without curl is singular

Earlier inverse norm approach:

$$\|\mathbf{f} + \nabla \bullet A\mathbf{U}\|_{-1}^2 + \|\mathbf{U} - \nabla\mathbf{u}\|_0^2$$

Identity-like in **U** & Poisson-like in **u**, but delicate & not λ-uniform

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IV. Norm Equivalence Motive

Consistency

$$F(\mathbf{U}^*;\mathbf{f}) = 0$$

Bilinearity

$$F(\mathbf{U};\mathbf{f}) = F(\mathbf{U}^* - \mathbf{U};\mathbf{0})$$

Norm equivalence

$$F(\mathbf{U}^* - \mathbf{U};\mathbf{0}) \sim \|\mathbf{U}^* - \mathbf{U}\|^2$$

$\Rightarrow$

$$\min_{\mathbf{U}^h \in \mathbf{S}^h} F(\mathbf{U}^h;\mathbf{f}) \sim \min_{\mathbf{U}^h \in \mathbf{S}^h} \left\|\mathbf{U}^* - \mathbf{U}^h\right\|^2$$

\+ fast solvers for diagonal Sobolev norms

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Norm Equivalence

$$tr(U,V,W,Z)^t = U + Z$$

$$F_1(\mathbf{U};\mathbf{0}) \sim \|\mathbf{U}\|_1^2 + \lambda^2 \|tr\mathbf{U}\|_1^2$$
$$\mathbf{U} \in H^1(\Omega)^4 \cap \mathbf{X}$$

$$F_0(\mathbf{U};\mathbf{0}) \sim \|\mathbf{U}\|_0^2 + \lambda^2 \|tr\mathbf{U}\|_0^2$$
$$\mathbf{U} \in H^{divA}(\Omega)^4 \cap \mathbf{X}$$

λ-uniform

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Standard Line of Proof

$A >> 0$

Upper bound :

by triangle inequality

Lower bound :

$$\mathbf{U} = \nabla \mathbf{v} + A^{-1} \nabla \times \mathbf{W}, \quad \nabla \bullet \mathbf{W} = \mathbf{0}$$

$$\|\nabla \bullet A\mathbf{U}\|^2 + \|\nabla \times \mathbf{U}\|^2 =$$
$$\|\nabla \bullet A \nabla \mathbf{v}\|^2 + \left\|\nabla \times A^{-1} \nabla \times \mathbf{W}\right\|^2$$

use regularity for $\mathbf{v}$ & $\mathbf{W}$

Elasticity Line of Proof

A singular

Lower bound :

$$\mathbf{U} = \nabla \mathbf{v} + A^{\dagger} \nabla \times \mathbf{W} + \begin{pmatrix} 0 \\ p \\ -p \\ 0 \end{pmatrix},$$
$$\nabla \bullet \mathbf{W} = \mathbf{0}$$

$$\|\nabla \bullet A\mathbf{U}\|^2 + \|\nabla \times \mathbf{U}\|^2 =$$
$$\|\nabla \bullet A \nabla \mathbf{v}\|^2 + \left\|\nabla \times A^{\dagger} \nabla \times \mathbf{W}\right\|^2 + \|\nabla p\|^2$$

- use elasticity regularity estimates for $\mathbf{v}$ & Stokes regularity estimates for $\mathbf{W}$ & p
- use care to be sure bounds are λ-uniform

V. Norm Deficiency

$$\|\mathbf{U}\|^2 + \lambda^2 \|tr\mathbf{U}\|^2$$

not a good separator for
U - components for large λ

$$\left\| \begin{pmatrix} U \\ 0 \\ 0 \\ \pm U \end{pmatrix} \right\|^2 + \lambda^2 \left\| tr \begin{pmatrix} U \\ 0 \\ 0 \\ \pm U \end{pmatrix} \right\|^2 = \begin{cases} (2+4\lambda^2)\|U\|^2 \\ 2\|U\|^2 \end{cases}$$

Rotation

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\tilde{\mathbf{U}} = Q\mathbf{U} = \begin{pmatrix} \frac{U+Z}{\sqrt{2}} \\ V \\ W \\ \frac{U-Z}{\sqrt{2}} \end{pmatrix} \quad \Rightarrow \quad \mathbf{U} = Q\tilde{\mathbf{U}}$$

$$\tilde{F}_1(\tilde{\mathbf{U}};\mathbf{f}) = \left\|\mathbf{f} + \nabla \bullet AQ\tilde{\mathbf{U}}\right\|_0^2 + \left\|\nabla \times Q\tilde{\mathbf{U}}\right\|_0^2$$
$$\sim \left\|\tilde{\mathbf{U}}\right\|_1^2 + \lambda^2 \left\|\tilde{U}\right\|_1^2$$

λ-uniform equivalence to diagonal norm

$\mathbf{FOSLS_1}$ Practice

everything is λ-uniform

- standard FE spaces for each variable & Rayleigh-Ritz yield *Poisson-like* estimates for displacement flux &, hence, deformations & stresses
- *simple multigrid methods using these FE spaces & Gauss-Seidel point relaxation* yield optimal performance
- displacements can be accurately & efficiently recovered by solving Poisson equations:

$$\min_{\mathbf{u}} \left\| \nabla \mathbf{u} - Q\tilde{\mathbf{U}} \right\|^2$$

$\mathbf{FOSLS_0}$ Practice

everything is λ-uniform

- standard FE spaces for each variable & Rayleigh-Ritz yield *identity-like* estimates for displacement flux &, hence, deformations & stresses
- *basic iterative methods like conjugate gradients, with scalar Poisson solvers for inverse norm evaluation,* yield optimal performance
- displacements can be accurately & efficiently recovered by solving Poisson equations:

$$\min_{\mathbf{u}} \left\| \nabla \mathbf{u} - Q\tilde{\mathbf{U}} \right\|^2$$

Error Estimators

For conventional energy functionals, the minimum value is unknown. For FOSLS, because the minimum is zero:

$$F(\mathbf{U};\mathbf{f}) = F(\mathbf{U}^* - \mathbf{U};\mathbf{0}) \sim \|\mathbf{U}^* - \mathbf{U}\|^2$$

&

$$F(\mathbf{O};\mathbf{f}) = F(\mathbf{U}^*;\mathbf{0}) \sim \|\mathbf{U}^*\|^2$$

$\Rightarrow \sqrt{F(\mathbf{U};\mathbf{f})}$ = absolute total error

&

$-\log_{10}\sqrt{F(\mathbf{U};\mathbf{f})/F(\mathbf{O};\mathbf{f})}$ = relative

no matter how **U** was obtained

Resources

Other methods:

'88	Brezzi & Douglas	Num. Math.
'88	Stenberg	Num. Math.
'89	Douglas & Wang	Math. Comp.
'91	Franca & Stenberg	SINUM
'93	Brenner	SINUM

no overall uniform performance estimates

FOSLS:

http://amath-www.colorado.edu/appm/

ftp://amath.colorado.edu/pub