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LINEARIZED INVERSE SCATTERING PROBLEMS IN ACOUSTICS AND ELASTICITY

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Using the single-scattering approximation we invert for the material parameters of an acoustic two-parameter medium and then for a three-parameter isotropic elastic medium. Our procedure is related to various methods of depth migration in seismics, i.e. methods for locating major discontinuities in the subsurface material without specifying which quantities are discontinuous or by how much they jump. Our asymptotic multiparameter inversion makes use of amplitude information to reconstruct the size of the jumps in the parameters describing the medium.

We allow spatially varying background parameters (both vertically and laterally) and an almost arbitrary source-receiver configuration. The computation is performed in the time domain and we use all available data even if it is redundant. This ability to incorporate the redundant information in a natural way is based upon a formula for double integrals over spheres.

We solve for perturbations in different parameters treating separately *P*-to-*P*, *P*-to-*S*, *S*-to-*P*, and *S*-to-*S* data. It turns out that one may invert using subsets of the data, or all of it together. We also describe modifications to our scheme which allow us to use the Kirchhoff instead of the Born approximation for the forward problem when the scatterers are smooth surfaces of discontinuity.

Introduction

In this paper we obtain an inversion and imaging procedure for finding the material parameters of the medium first for an acoustic two-parameter medium (specific volume and compressibility) and then, using the same framework, for the three-parameter isotropic elastic medium (density and Lamé constants).

An expanded abstract describing the results obtained during the summer of 1986 has already appeared separately [1]. In this paper we are concerned with the mathematical analysis of the problem. We linearize the inverse scattering problem, which is the simplest way of treating many practical problems in seismic exploration, medical imaging, nondestructive testing, and other applications. Linearization of the inverse scattering problem is achieved by considering the actual medium as a perturbation of a (spatially varying) background model. Then by using the single scattering (Born) approximation we obtain integral equations relating the singly scattered field linearly to the unknown parameters.

Asymptotic solutions of the linearized inverse problem have been rigorously derived using the theory of the generalized Radon transform [2, 3]. The linearized inverse problem was analysed from this point of view for the Helmholtz equation by Beylkin [4], Miller et al. [5, 6], and Beylkin et al. [7]. In this paper we extend the analysis to an acoustic medium with two parameters and then to an isotropic elastic solid with three parameters. For the Helmholtz equation a rigorous treatment of the linearized inverse problem yields a correct treatment of amplitude information. The multiparameter inversion described in this paper makes additional use of amplitude information to reconstruct the several parameters describing the medium.

The method we use for inverting seismic data is closely related to various methods of migration, by which one constructs an image of the geological structure below seismic lines (of sources and receivers) and which position features correctly. However, a migration algorithm is a method for locating major discontinuities in the subsurface material without specifying which quantities are discontinuous or by how much they jump [8, 9]. As such they go back to Hagedoorn [10] and beyond (see [11] and the recent review Stolt and Weglein [12]).

At about the same time as early migration schemes were being developed a precise mathematical theory to invert quantum physical scattering data was proposed by Gelfand and Levitan [13] and Marchenko [14] for one-dimensional problems. Since these beginnings two dissimilar lines of research have developed which are now converging. On the one hand there is the purely one-dimensional inversion generalized to obtain more than one parameter: see Stickler [15], who follows an approach of Deift and Trubowitz [16], and Coen [17]. Generalization to higher dimensions is possible to some extent, and, at the cost of linearization, to a much greater extent: see Cohen and Bleistein [18] and Bleistein and Cohen [19]. On the other hand, after the tradition of seismic migration started by Claerbout [20, 21], Stolt [22] introduced a method of migration by Fourier transforms, and Clayton and Stolt [23] developed a related method of inverting seismic data for more than one parameter. Integral approaches to migration were described by French [24, 25] and Schneider [26]. Norton [27] inverted for two parameters simultaneously by using single frequency data for source–receiver pairs having just two scattering angles. Multiparameter inversion in the context of single frequency diffraction tomography was proposed by Devaney [28]. A linearized inverse problem with constant background parameters was treated by Boyse and Keller [29].

This paper differs in a number of ways from previous work. First, we allow spatially varying background parameters (both vertically and laterally) and an almost arbitrary source–receiver configuration. Second, the actual computation is performed in the time domain, which removes certain difficulties associated with the use of the frequency domain as, for example, in Clayton and Stolt [23]. Third, we use all available data even if it is redundant. It is our belief that, with data as imperfect as in practice it always is, no data should be neglected merely because of redundancy. This ability to incorporate the redundant information in a natural way is based upon a formula for double integrals over spheres [30].

We start with the analysis of the acoustic case and linearize the inverse scattering problem in a nonhomogeneous fluid characterized by two parameters: specific volume (reciprocal of density) and compressibility (reciprocal of bulk modulus). In the process of formulating the linearized inverse scattering problem we identify the combinations of the parameters which have physical significance and can be reconstructed. These combinations appear naturally within our method.

In Section 2 we extend our analysis to the elastic case where the field quantity is the elastic displacement vector which satisfies the elastic wave equation. Since the elastic equation carries longitudinal (P) waves as well as transverse (S) waves it is necessary to consider P-to-P, P-to-S, S-to-P, and S-to-S scattering. This is made more complicated by the fact that S-waves have two independent polarizations. The calculations are carried as far as setting up the linearized inverse problem for the various combinations of the unknown coefficients. These combinations (amplitude radiation patterns) contain angular dependent coefficients similar to those derived by Wu and Aki [31]. The only new element here beyond the acoustic case is the presence of several different scattering modes, such as P-to-P, P-to-S, S-to-P, and, because S-waves are polarized, two S-to-S modes.

Both acoustic and elastic linearized inverse problems can be reduced to the problem of inversion of a generalized Radon transform. In Section 3 we define this transform and solve the inversion problem asymptotically with respect to smoothness; we retain only the most singular (discontinuous) term, and this allows us to recover the size of the jump in the case of jump discontinuities in the parameters. The reconstruction procedure itself amounts to several generalized backprojections followed by the solution of a small linear system at each point of reconstruction.

In Section 4 we apply the inversion procedure for the generalized Radon transform to solve the linearized inverse scattering problem for acoustic media. We solve for perturbations in two parameters—specific volume and compressibility, or rather the logarithmic derivatives of these quantities. The reconstruction procedure calls for asymptotic information about the Green's function of the background medium, specifically its phase (travel time) and amplitude, which opens up the possibility of computing by ray methods. This seems particularly advantageous in large-scale problems in three space dimensions. The formulas for inversion are quite simple and lend themselves to robust practical implementations. We demonstrate this with a numerical example. We describe our experiences when the algorithm is applied to scalar data generated by a finite-difference code with a fairly complicated model.

In Section 5 we apply the inversion procedure for the generalized Radon transform of Section 3 to solve the linearized inverse problem for elastic media. We solve for perturbations in different parameters treating separately P-to-P, P-to-S, S-to-P, and S-to-S data. It turns out that one may invert using subsets of the data, or all of it together. We describe the procedures for inversion of the individual scattering modes and the combinations of the parameters that can be recovered. We do not treat the problem of separating the different modes of scattering in the elastic case in detail, though our inversion procedure itself accomplishes this at least partially. First, it requires an ordinary projection of the multicomponent data on the unit vectors associated with ray directions within the background medium. We note that this usually is the foundation of algorithms for P and S separation (see [32], for example). Second, generalized backprojection itself acts as a filter and will tend to suppress modes which are not being used.

In Section 6 we describe modifications to our scheme which allow us to use the Kirchhoff approximation as an approximation for the forward problem. The difference between the images of parameters obtained under the assumption of the Kirchhoff approximation and the Born approximation appears to be equivalent to the effect of a spatial filter and does not affect the recovery of the parameters as such. Recently, Bleistein [33] proposed a modification of the algorithms described in Beylkin [4] and Miller et al. [5, 6]. The modification is based upon the use of the Kirchhoff instead of the Born approximation in the forward problem and results in reconstruction of the reflection coefficient as a function of angle. Parsons [34] used this scheme where he linearized the reflection coefficient to obtain the parameters. In Section 6 we also discuss the differences between our approach and those in [33] and [34].

The method we propose has advantages over other approaches suggested recently. Specifically, the so-called nonlinear inversions proposed by a number of authors (Tarantola [35], Mora [36]) in fact iterate the solution of the linearized inverse problem. This, of course, can be done with our method as well. So it makes sense to compare our method with a single step of such an iterative algorithm. One can see that our method allows us to quantify the size of the recovered discontinuities, while, as far as we know, the methods mentioned above would require several iterations to achieve the same result.

In summary our method is *equivalent* to the following four steps (although it is not implemented in this way):

1. Given the background model, for every source s, receiver r and image point y, we define θ to be the angle at y between the rays connecting y to s and y to r.

2. For each point y collect all source–receiver pairs with the *same* θ into what might be called constant θ gathers.

3. For fixed θ solve the integral equations such as (1.21), (2.49), (2.56), etc. for the (θ-dependent) linear combinations of parameters in (1.20), (2.48), (2.55), etc. This step uses the generalized Radon transform inversion formula and is repeated for many separate values of θ.

4. Solve for the unknown parameters using generalized linear inversion for over-determined systems.

In implementing this procedure the sums or integrals involved in the inverse Radon transform of step 3 and the generalized linear inverse of step 4 are combined so that the constant θ gathers mentioned in step 2 are never explicitly formed. In fact the data may be read just once and in an arbitrary order.

1. Linearized scattering for acoustics

An acoustic medium (a nonhomogeneous fluid) is characterized by specific volume σ (reciprocal of density) and compressibility (reciprocal of bulk modulus) $\kappa = \sigma/c^2$, where c is the sound speed. The pressure p in the absence of sources satisfies the equation

$$\kappa(x)\,\partial_t^2 p(s, x, t) - (\sigma(x)\, p_{,j}(s, x, t))_{,j} = 0. \tag{1.1}$$

Here t is time, ∂_t^2 is the second time derivative and $p_{,j}$ denotes $\partial p/\partial x_j$. In (1.1) summation is implied over repeated indices. We consider a three-dimensional medium, and so index $j = 1, 2, 3$ and $x = (x_1, x_2, x_3)$ is the cartesian position vector. However, all the derivations can be carried out for an arbitrary dimensionality.

Let specific volume σ and compressibility κ be written as

$$\sigma(x) = \sigma^0(x) + \sigma'(x), \qquad \kappa(x) = \kappa^0(x) + \kappa'(x). \tag{1.2-1.3}$$

Here σ^0 and κ^0 are known *background* specific volume and compressibility, and σ' and κ' the corresponding perturbations, the latter having bounded supports.

The inverse problem we wish to solve is to find the perturbations κ'/κ^0 and σ'/σ^0 as functions of position using observations of the singly scattered field. We assume that the sources s and receivers r lie on a closed surface ∂D bounding a domain D which contains the supports of σ' and κ' and contained within the region of validity of (1.1). If, however, ∂D coincides with a physical boundary on which boundary conditions may be specified some modification in the derivation of the integral representation of the singly scattered field is necessary. This is discussed in Appendix A.

For fixed s let $\tilde{G} = \tilde{G}(s, x, t)$ be the incident field which, as a function of x, solves the following equation

$$\kappa^0 \partial_t^2 \tilde{G} - (\sigma^0 \tilde{G}_{,j})_{,j} = \delta(t)\,\delta(x - s), \qquad \tilde{G}|_{t<0} = 0. \tag{1.4-1.5}$$

Within the single scattering approximation the scattered field $U = U(x, s, t)$ (due to the incident field $\tilde{G}$) is the solution of the equation

$$\kappa^0 \partial_t^2 U - (\sigma^0 U_{,j})_{,j} = -\kappa' \partial_t^2 \tilde{G} + (\sigma' \tilde{G}_{,j})_{,j}. \tag{1.6}$$

Let $\hat{G} = \hat{G}(x, r, t)$ be the solution of the equation

$$\kappa^0 \partial_t^2 \hat{G} - (\sigma^0 \hat{G}_{,j})_{,j} = \delta(t)\,\delta(x - r), \qquad \hat{G}|_{t<0} = 0, \tag{1.7-1.8}$$

where r denotes the receiver position.

Since $\hat{G}$ is the Green's function for eq. (1.6) we arrive at

$$U(s, r, t) = \int_D \mathrm{d}x\,[-\kappa' \partial_t^2 \tilde{G} + (\sigma' \tilde{G}_{,j})_{,j}] *_t \hat{G}, \tag{1.9}$$

where $*_t$ denotes convolution in time. Integrating the second term on the right by parts and noticing that the boundary term vanishes since the perturbation σ' is zero on ∂D, we obtain

$$U(s, r, t) = -\int_D \mathrm{d}x\,[\kappa' \partial_t^2 \tilde{G} *_t \hat{G} + \sigma' \tilde{G}_{,j} *_t \hat{G}_{,j}], \tag{1.10}$$

which is the integral representation of the singly scattered field.

Here and throughout the paper we will consider the most singular term of the integral representations of the singly scattered fields. However, by making this approximation in the direct problem we are not making an additional approximation with respect to the inverse problem. This is because we proceed to solve the inverse problem *modulo a smooth error.* To obtain such a solution it is sufficient to consider only the most singular term in the direct problem. Indeed, only this term contributes to the most singular term that is recovered. This observation is discussed in greater detail in Appendix B.

To obtain the most singular term of the singly scattered field we consider the most singular part of the Green's function and its derivatives:

$$\tilde{G}(s,x,t)\approx\tilde{A}(s,x)\,\delta(t-\tilde{\phi}(s,x)),\qquad \hat{G}(x,r,t)\approx\hat{A}(x,r)\,\delta(t-\hat{\phi}(x,r)),\tag{1.11-1.12}$$

$$\tilde{G}_{,j}(s,x,t)\approx-\tilde{A}(s,x)\,\tilde{\phi}_{,j}(s,x)\,\delta'(t-\tilde{\phi}(s,x)),\tag{1.13}$$

$$\hat{G}_{,j}(x,r,t)\approx-\hat{A}(x,r)\,\hat{\phi}_{,j}(x,r)\,\delta'(t-\hat{\phi}(x,r)).\tag{1.14}$$

Substituting (1.11)-(1.14) in (1.10) we arrive at

$$U(s,r,t)=-\partial_t^2\int_D \mathrm{d}x\,[\kappa'+\sigma'\tilde{\phi}_{,j}\hat{\phi}_{,j}]\tilde{A}\hat{A}\,\delta(t-\tilde{\phi}-\hat{\phi}).\tag{1.15}$$

Here the phase functions $\tilde{\phi}=\tilde{\phi}(s,x)$ and $\hat{\phi}=\hat{\phi}(x,r)$ satisfy the eikonal equations with the background sound speed $c^0=\sqrt{\sigma^0/\kappa^0}$

$$\sum_j[\tilde{\phi}_{,j}(s,x)]^2=\frac{\kappa^0(x)}{\sigma^0(x)},\qquad \sum_j[\hat{\phi}_{,j}(x,r)]^2=\frac{\kappa^0(x)}{\sigma^0(x)}.\tag{1.16}$$

For fixed x, s and r, the functions $\tilde{\phi}(s,x)$ and $\hat{\phi}(x,r)$ are travel times from s to x, and from x to r.

Amplitudes $\tilde{A}=\tilde{A}(s,x)$ and $\hat{A}=\hat{A}(x,r)$ in (1.15) satisfy the transport equations

$$(\sigma^0(x)\,\tilde{A}^2(s,x)\,\tilde{\phi}_{,j}(s,x))_{,j}=0,\qquad (\sigma^0(x)\,\hat{A}^2(x,r)\,\hat{\phi}_{,j}(x,r))_{,j}=0.\tag{1.17-1.18}$$

These equations describe changes of the amplitude along rays connecting s with x, and x with r. The transport equations reduce to ordinary differential equations along these rays and $\tilde{A}$ and $\hat{A}$ are determined uniquely by initial conditions which depend on the choice of the Green's function and, therefore, on local conditions in the vicinity of the source and the receiver (see Appendix A). If the background sound speed c^0 is discontinuous then the rays satisfy Snell's law on the surfaces of discontinuity and appropriate transmission coefficients have to be used in computing amplitudes.

Using the eikonal equations (1.16) we can write the inner product $\tilde{\phi}_{,j}\hat{\phi}_{,j}$ as

$$\tilde{\phi}_{,j}\hat{\phi}_{,j}=\frac{\kappa^0}{\sigma^0}\cos\theta,\tag{1.19}$$

where $\theta=\theta(x,s,r)$ is the angle at x between the directions of rays connecting x with s and with r. Using (1.19), (1.2) and (1.3) we write the expression in brackets in (1.15) as follows

$$\kappa'+\sigma'\tilde{\phi}_{,j}\hat{\phi}_{,j}=\kappa^0\left[\frac{\kappa'}{\kappa^0}+\frac{\sigma'}{\sigma^0}\cos\theta\right]\equiv\kappa^0 f(x,\theta),\tag{1.20}$$

say. Using (1.20) in (1.15) we finally obtain the expression for the most singular term of the singly scattered field:

$$U(s,r,t)=-\partial_t^2\int_D \mathrm{d}x\,f(x,\theta)\,\kappa^0(x)\,\tilde{A}(s,x)\,\hat{A}(x,r)\,\delta(t-\tilde{\phi}(s,x)-\hat{\phi}(x,r)).\tag{1.21}$$

So far our analysis has concerned the direct scattering problem. It is our goal, however, to consider (1.21) not as an expression for $U(s, r, t)$ but as an integral equation for f, where U is known. We recall that U is the scattered field and does not contain the unperturbed field due to the background material itself. This scattered field should be separated from the full response, which usually is a straightforward matter and is not considered here.

The integral in (1.21) has the form of a generalized Radon transform of the kind considered in [2, 3] (see also Section 3 of this paper) and so we can use the inversion formulas derived there to solve the integral equation for $f(x, \theta)$ for many fixed values of θ. Using the form of the amplitude radiation pattern in (1.20) we may then find κ'/κ^0 and σ'/σ^0 separately. Since we have $f(x, \theta)$ for many values of θ this latter step can be treated as a generalized linear inversion. In practice we perform this generalized linear inversion and the inversion of the generalized Radon transform together (see Section 3). Inversion of (1.21) is carried out in Section 4.

In the next section we derive formulas analogous to (1.21) for elastic wave scattering. The inversion of these formulas is carried out in Section 5.

2. Linearized scattering for elastic solid

Wave propagation in an inhomogeneous anisotropic elastic solid in the absence of sources is governed by the equations

$$\rho \partial_t^2 u_l - (c_{lmpq} u_{p,q})_{,m} = 0, \tag{2.1}$$

where $u_l = u_l(x, t)$ is the l-component of the displacement vector, $\rho = \rho(x)$ is the density and $c_{lmpq} = c_{lmpq}(x)$ are the elastic constants of the medium at the point $x = (x_1, x_2, x_3)$. The elastic constants satisfy the symmetry relations $c_{lmpq} = c_{mlpq} = c_{pqlm}$. For an isotropic solid

$$c_{lmpq} = \lambda \delta_{lm} \delta_{pq} + \mu(\delta_{lp}\delta_{mq} + \delta_{lq}\delta_{mp}), \tag{2.2}$$

where λ and μ are the Lamé constants of the medium and δ_{lm} is the Kronecker symbol.

Suppose the density and Lamé constants can be written as

$$\rho = \rho^0 + \rho', \qquad \lambda = \lambda^0 + \lambda', \qquad \mu = \mu^0 + \mu', \tag{2.3-2.5}$$

where ρ^0, λ^0 and μ^0 are known *background* density and Lamé constants, and ρ', λ' and μ' the corresponding perturbations, the latter having their compact supports strictly inside some domain D.

As in the previous section we assume that the boundary ∂D is not a physical boundary but merely bounds a subdomain D of the region where (2.1) holds. We discuss modifications required if the boundary ∂D is a physical boundary (on which one may specify boundary conditions) in Appendix A.

Let $\tilde{G}_{jl} = \tilde{G}_{jl}(s, x, t)$ be the incident field, which satisfies the following equations

$$\rho^0 \partial_t^2 \tilde{G}_{jl} - (c^0_{lmpq} \tilde{G}_{jp,q})_{,m} = \delta_{jl}\delta(t)\delta(x-s), \qquad \tilde{G}_{jl}|_{t<0} = 0. \tag{2.6-2.7}$$

Here $\tilde{G}_{jl}$ is the displacement in the l-direction at the point x due to a point force in the j-direction at the point s. Within the single-scattering approximation the scattered field $U_{jl} = U_{jl}(s, x, t)$ is the solution of the equation

$$\rho^0 \partial_t^2 U_{jl} - (c^0_{lmpq} U_{jp,q})_{,m} = -\rho' \partial_t^2 \tilde{G}_{jl} + (c'_{lmpq} \tilde{G}_{jp,q})_{,m}. \tag{2.8}$$

Let $\hat{G}_{kl} = \hat{G}_{kl}(x, r, t)$ satisfy the following equations

$$\rho^0 \partial_t^2 \hat{G}_{kl} - (c^0_{lmpq} \hat{G}_{kp,q})_{,m} = \delta_{kl}\delta(t)\delta(x-r), \qquad \hat{G}_{kl}|_{t<0} = 0, \tag{2.9-2.10}$$

where r denotes the receiver position.

Then $\hat{G}_{kl}$ is the Green's function for eq. (2.8) and we arrive at

$$U_{jk}(s, r, t) = -\int_D [\rho' \partial_t^2 \tilde{G}_{jl} *_t \hat{G}_{kl} - (c'_{lmpq} \tilde{G}_{jp,q})_{,m} *_t \hat{G}_{kl}]\, \mathrm{d}x. \tag{2.11}$$

Integrating the second term on the right by parts and noticing that the boundary term vanishes because the perturbations c'_{lmpq} are zero on the boundary ∂D, we obtain

$$U_{jk}(s, r, t) = -\int_D [\rho' \partial_t^2 \tilde{G}_{jl} *_t \hat{G}_{kl} + c'_{lmpq} \tilde{G}_{jp,q} *_t \hat{G}_{kl,m}]\, \mathrm{d}x, \tag{2.12}$$

which is an integral representation of the singly scattered field. Here U_{jk} is the k-component of the scattered field at the receiver location r due to the point force in the j-direction at the source location s.

Isotropic solid. As we mentioned before (see also Appendix B) for inversion of the most singular part of the perturbation it is sufficient to consider the most singular term of the integral representation of the singly scattered field (2.12). This term can be obtained by using the most singular part of the Green's functions in (2.6) and (2.9). Thus, $\tilde{G}_{jl}$ is a superposition of several terms like

$$\tilde{G}_{jl} \approx \tilde{A}_{jl}\delta(t - \tilde{\phi}), \tag{2.13}$$

one for P waves and possibly two for S. On substituting (2.13) in (2.6) we arrive at a set of equations for the phase $\tilde{\phi}$ and vector amplitudes $\tilde{A}^k_{jl}$. In the general anisotropic case (see e.g. Burridge [37]) the phase function $\tilde{\phi}$ is determined by the condition

$$\det(\rho^0 \delta_{lp} - c^0_{lmpq} \tilde{\phi}_{,m} \tilde{\phi}_{,q}) = 0, \tag{2.14}$$

and the amplitude of the main term satisfies

$$(\rho^0 \delta_{lp} - c^0_{lmpq} \tilde{\phi}_{,m} \tilde{\phi}_{,q}) \tilde{A}_{jp} = 0, \tag{2.15}$$

$$(c^0_{lmpq} \tilde{A}_{jl} \tilde{A}_{jp} \tilde{\phi}_{,q})_{,m} = 0, \qquad \text{no summation over } j. \tag{2.16}$$

The most singular part of the Green's function is a sum of terms of the form in (2.13) where phases and amplitudes satisfy (2.14)-(2.16). The same procedure applies to the Green's function in (2.9)-(2.10).

For an isotropic solid the leading singular term of the Green's functions in (2.6)-(2.7) and (2.9)-(2.10) can be written in the form

$$\tilde{G}_{jl} = \tilde{G}^P_{jl} + \tilde{G}^S_{jl}, \qquad \hat{G}_{kl} = \hat{G}^P_{kl} + \hat{G}^S_{kl}, \tag{2.17-2.18}$$

where

$$\tilde{G}^P_{jl} = \tilde{A}^P_{jl}\delta(t - \tilde{\phi}^P), \quad \tilde{G}^S_{jl} = \tilde{A}^S_{jl}\delta(t - \tilde{\phi}^S), \quad \hat{G}^P_{kl} = \hat{A}^P_{kl}\delta(t - \hat{\phi}^P), \quad \hat{G}^S_{kl} = \hat{A}^S_{kl}\delta(t - \hat{\phi}^S). \tag{2.19-2.22}$$

The leading singular terms of the spatial derivatives of the Green's functions are as follows

$$\tilde{G}^P_{jl,p} = -\tilde{\phi}^P_{,p} \tilde{A}^P_{jl} \delta'(t - \tilde{\phi}^P), \qquad \tilde{G}^S_{jl,p} = -\tilde{\phi}^S_{,p} \tilde{A}^S_{jl} \delta'(t - \tilde{\phi}^S), \tag{2.23-2.24}$$

$$\hat{G}^P_{kl,p} = -\hat{\phi}^P_{,p} \hat{A}^P_{kl} \delta'(t - \hat{\phi}^P), \qquad \hat{G}^S_{kl,p} = -\hat{\phi}^S_{,p} \hat{A}^S_{kl} \delta'(t - \hat{\phi}^S). \tag{2.25-2.26}$$

Equation (2.14) for the phase functions simplifies and the phase functions $\tilde{\phi}^P$, $\hat{\phi}^P$, $\tilde{\phi}^S$ and $\hat{\phi}^S$ satisfy the eikonal equations

$$\sum_{j=1,2,3}[\tilde{\phi}^P_{,j}]^2=\sum_{k=1,2,3}[\hat{\phi}^P_{,k}]^2=\frac{1}{c_P^2}, \tag{2.27}$$

$$\sum_{j=1,2,3}[\tilde{\phi}^S_{,j}]^2=\sum_{k=1,2,3}[\hat{\phi}^S_{,k}]^2=\frac{1}{c_S^2}, \tag{2.28}$$

where $c_P(x)=\sqrt{(\lambda^0(x)+2\mu^0(x))/\rho^0(x)}$ and $c_S(x)=\sqrt{\mu^0(x)/\rho^0(x)}$ are the P-wave speed and the S-wave speed. Here $\rho^0(x)$, $\lambda^0(x)$ and $\mu^0(x)$ are the density and elastic constants for the background isotropic elastic medium.

The amplitudes $\tilde{A}^P_{jl}$, $\hat{A}^P_{kl}$, $\tilde{A}^S_{jl}$ and $\hat{A}^S_{kl}$ satisfy transport equations which can be written as

$$(\rho^0c_P^2\tilde{A}^P_{jp}\tilde{A}^P_{jp}\tilde{\phi}^P_{,m})_{,m}=0, \qquad \text{no summation over } j, \tag{2.29}$$

$$(\rho^0c_P^2\hat{A}^P_{kp}\hat{A}^P_{kp}\hat{\phi}^P_{,m})_{,m}=0, \qquad \text{no summation over } k, \tag{2.30}$$

$$(\rho^0c_S^2\tilde{A}^S_{jp}\tilde{A}^S_{jp}\tilde{\phi}^S_{,m})_{,m}=0, \qquad \text{no summation over } j, \tag{2.31}$$

$$(\rho^0c_S^2\hat{A}^S_{kp}\hat{A}^S_{kp}\hat{\phi}^S_{,m})_{,m}=0, \qquad \text{no summation over } k. \tag{2.32}$$

We obtain the high frequency approximation of the integral representation of the singly scattered field by substituting (2.17)-(2.26) into (2.12). Then

$$U_{jk}(s,r,t)=U^{PP}_{jk}(s,r,t)+U^{PS}_{jk}(s,r,t)+U^{SP}_{jk}(s,r,t)+U^{SS}_{jk}(s,r,t), \tag{2.33}$$

where

$$U^{PP}_{jk}=-\partial_t^2\int_D[\rho'\delta_{pl}+c'_{lmpq}\tilde{\phi}^P_{,q}\hat{\phi}^P_{,m}]\tilde{A}^P_{jp}\hat{A}^P_{kl}\,\delta(t-\tilde{\phi}^P-\hat{\phi}^P)\,\mathrm{d}x, \tag{2.34}$$

$$U^{PS}_{jk}=-\partial_t^2\int_D[\rho'\delta_{pl}+c'_{lmpq}\tilde{\phi}^P_{,q}\hat{\phi}^S_{,m}]\tilde{A}^P_{jp}\hat{A}^S_{kl}\,\delta(t-\tilde{\phi}^P-\hat{\phi}^S)\,\mathrm{d}x, \tag{2.35}$$

$$U^{SP}_{jk}=-\partial_t^2\int_D[\rho'\delta_{pl}+c'_{lmpq}\tilde{\phi}^S_{,q}\hat{\phi}^P_{,m}]\tilde{A}^S_{jp}\hat{A}^P_{kl}\,\delta(t-\tilde{\phi}^S-\hat{\phi}^P)\,\mathrm{d}x, \tag{2.36}$$

$$U^{SS}_{jk}=-\partial_t^2\int_D[\rho'\delta_{pl}+c'_{lmpq}\tilde{\phi}^S_{,q}\hat{\phi}^S_{,m}]\tilde{A}^S_{jp}\hat{A}^S_{kl}\,\delta(t-\tilde{\phi}^S-\hat{\phi}^S)\,\mathrm{d}x. \tag{2.37}$$

We introduce the unit vectors tangent to the rays at the point x, denoting them by $\tilde{\alpha}^P=(\tilde{\alpha}^P_1,\tilde{\alpha}^P_2,\tilde{\alpha}^P_3)$, $\tilde{\alpha}^S=(\tilde{\alpha}^S_1,\tilde{\alpha}^S_2,\tilde{\alpha}^S_3)$, $\hat{\alpha}^P=(\hat{\alpha}^P_1,\hat{\alpha}^P_2,\hat{\alpha}^P_3)$ and $\hat{\alpha}^S=(\hat{\alpha}^S_1,\hat{\alpha}^S_2,\hat{\alpha}^S_3)$. Then

$$\tilde{\alpha}^P_j=c_P\tilde{\phi}^P_{,j}, \qquad \tilde{\alpha}^S_j=c_S\tilde{\phi}^S_{,j}, \qquad \hat{\alpha}^P_j=c_P\hat{\phi}^P_{,j}, \qquad \hat{\alpha}^S_j=c_S\hat{\phi}^S_{,j}, \tag{2.38-2.41}$$

where $j=1,2,3$ and we have used (2.27) and (2.28). Let us introduce angles $\theta^{PP}=\theta^{PP}(x,s,r)$, $\theta^{PS}=\theta^{PS}(x,s,r)$, $\theta^{SP}=\theta^{SP}(x,s,r)$ and $\theta^{SS}=\theta^{SS}(x,s,r)$ between pairs of unit vectors $\tilde{\alpha}^P$ and $\hat{\alpha}^P$, $\tilde{\alpha}^P$ and $\hat{\alpha}^S$, $\tilde{\alpha}^S$ and $\hat{\alpha}^P$, and $\tilde{\alpha}^S$ and $\hat{\alpha}^S$ respectively. The following relations hold

$$\cos\theta^{PP}=\tilde{\alpha}^P_j\hat{\alpha}^P_j, \qquad \cos\theta^{PS}=\tilde{\alpha}^P_j\hat{\alpha}^S_j, \qquad \cos\theta^{SP}=\tilde{\alpha}^S_j\hat{\alpha}^P_j, \qquad \cos\theta^{SS}=\tilde{\alpha}^S_j\hat{\alpha}^S_j. \tag{2.42-2.45}$$

We now rewrite the equations (2.34)-(2.37).

P-to-P scattering. The amplitudes in formula (2.34) can be written in the form $\tilde{A}^P_{jp} = \tilde{\alpha}^P_p \tilde{A}^P_j$ and $\hat{A}^P_{kl} = \hat{\alpha}^P_l \hat{A}^P_k$, where it follows from (2.29) and (2.30) that $\tilde{A}^P_j$ and $\hat{A}^P_k$ satisfy the equations

$$(\rho^0 c_P^2 \tilde{A}^P_j \tilde{A}^P_j \tilde{\phi}^P_{,m})_{,m} = 0, \qquad \text{no summation over } j, \tag{2.46}$$

$$(\rho^0 c_P^2 \hat{A}^P_k \hat{A}^P_k \hat{\phi}^P_{,m})_{,m} = 0, \qquad \text{no summation over } k. \tag{2.47}$$

Using (2.2)–(2.5), (2.38), (2.40) and (2.42) we have

$$[\rho'\delta_{pl} + c'_{lmpq}\tilde{\phi}^P_{,q}\hat{\phi}^P_{,m}]\tilde{\alpha}^P_p\hat{\alpha}^P_l = \rho^0\left[\frac{\lambda'}{\lambda^0+2\mu^0} + \frac{\rho'}{\rho^0}\cos\theta^{PP} + \frac{2\mu'}{\lambda^0+2\mu^0}\cos^2\theta^{PP}\right], \tag{2.48}$$

and, therefore (2.34) can be rewritten as follows.

$$U^{PP}_{jk} = -\partial_t^2 \int_D \rho^0\left[\frac{\lambda'}{\lambda^0+2\mu^0} + \frac{\rho'}{\rho^0}\cos\theta^{PP} + \frac{2\mu'}{\lambda^0+2\mu^0}\cos^2\theta^{PP}\right]\tilde{A}^P_j\hat{A}^P_k\delta(t-\tilde{\phi}^P-\hat{\phi}^P)\,\mathrm{d}x. \tag{2.49}$$

P-to-S scattering. In this case in addition we need the unit vector $\gamma^{PS} = (\gamma^{PS}_1, \gamma^{PS}_2, \gamma^{PS}_3)$ in the direction of $\tilde{\alpha}^P \times \hat{\alpha}^S$, as well as the unit vector $\hat{\beta}^{PS} = (\hat{\beta}^{PS}_1, \hat{\beta}^{PS}_2, \hat{\beta}^{PS}_3)$ orthogonal to both $\hat{\alpha}^S$ and γ^{PS}. We have

$$\tilde{\alpha}^P_j\hat{\beta}^{PS}_j = \sin\theta^{PS}, \qquad \tilde{\alpha}^P_j\gamma^{PS}_j = 0. \tag{2.50-2.51}$$

The amplitude $\hat{A}^S_{kl}$ of the S-wave in formula (2.32) can be written in the form

$$\hat{A}^S_{kl} = \hat{A}^{PS}_k(\hat{\beta}^{PS}_l\hat{a}^{PS}_1 + \gamma^{PS}_l\hat{a}^{PS}_2), \tag{2.52}$$

where $(\hat{a}^{PS}_1)^2 + (\hat{a}^{PS}_2)^2 = 1$ and $\hat{A}^{PS}_k$ satisfies the transport equation

$$(\rho^0 c_S^2 \hat{A}^{PS}_k \hat{A}^{PS}_k \hat{\phi}^S_{,m})_{,m} = 0, \qquad \text{no summation over } k. \tag{2.53}$$

It follows from (2.51) and (2.52) that

$$\hat{A}^S_{kl}\hat{\beta}^{PS}_l = \hat{A}^{PS}_k\hat{a}^{PS}_1. \tag{2.54}$$

Using (2.2), (2.38), (2.40), (2.50), (2.51) and (2.43) we have

$$[\rho'\delta_{pl} + c'_{lmpq}\tilde{\phi}^P_{,q}\hat{\phi}^S_{,m}]\tilde{\alpha}^P_p(\hat{\beta}^{PS}_l\hat{a}^{PS}_1 + \gamma^{PS}_l\hat{a}^{PS}_2) = \rho^0\left[\frac{\rho'}{\rho^0}\sin\theta^{PS} + \frac{\mu'}{\mu^0}\frac{c_S}{c_P}\sin 2\theta^{PS}\right]\hat{a}^{PS}_1, \tag{2.55}$$

and, therefore (2.35) can be rewritten as follows

$$U^{PS}_{jk} = -\partial_t^2 \int_D \rho^0\left[\frac{\rho'}{\rho^0}\sin\theta^{PS} + \frac{\mu'}{\mu^0}\frac{c_S}{c_P}\sin 2\theta^{PS}\right]\tilde{A}^P_j\hat{A}^S_{kl}\hat{\beta}^{PS}_l\delta(t-\tilde{\phi}^P-\hat{\phi}^S)\,\mathrm{d}x, \tag{2.56}$$

where we used (2.54) to obtain the amplitude term.

S-to-P scattering. This case is similar to the previous one. We consider the unit vector $\gamma^{SP} = (\gamma^{SP}_1, \gamma^{SP}_2, \gamma^{SP}_3)$ in the direction of $\tilde{\alpha}^S \times \hat{\alpha}^P$ and the unit vector $\tilde{\beta}^{SP} = (\tilde{\beta}^{SP}_1, \tilde{\beta}^{SP}_2, \tilde{\beta}^{SP}_3)$ orthogonal to both $\tilde{\alpha}^S$ and γ^{SP}. We have

$$\tilde{\beta}^{SP}_j\hat{\alpha}^P_j = -\sin\theta^{SP}, \qquad \gamma^{SP}_j\hat{\alpha}^P_j = 0. \tag{2.57-2.58}$$

The amplitude $\tilde{A}^S_{jp}$ of the S-wave in formula (2.31) can be written in the form

$$\tilde{A}^S_{jp} = \tilde{A}^{SP}_j(\tilde{\beta}^{SP}_p\tilde{a}^{SP}_1 + \gamma^{SP}_p\tilde{a}^{SP}_2), \tag{2.59}$$

$(\tilde{a}_1^{SP})^2+(\tilde{a}_2^{SP})^2=1$ and $\tilde{A}_j^{SP}$ satisfies the transport equation

$$(\rho^0 c_S^2 \tilde{A}_j^{SP}\tilde{A}_j^{SP}\tilde{\phi}^S_{,m})_{,m}=0, \qquad \text{no summation over } j. \tag{2.60}$$

We also have

$$\tilde{A}^S_{jp}\tilde{\beta}^{SP}_p=\tilde{A}_j^{SP}\tilde{a}_1^{SP}. \tag{2.61}$$

We rewrite the integrand of eq. (2.36) in a form similar to (2.55) to obtain

$$U_{jk}^{SP}=\partial_t^2\int_D \rho^0\left[\frac{\rho'}{\rho^0}\sin\theta^{SP}+\frac{\mu'}{\mu^0}\frac{c_S}{c_P}\sin 2\theta^{SP}\right]\tilde{A}^S_{jp}\tilde{\beta}^{SP}_p\hat{A}^P_k\delta(t-\tilde{\phi}^S-\hat{\phi}^P)\,\mathrm{d}x. \tag{2.62}$$

S-to-S scattering. In this case we need the unit vector $\gamma^{SS}=(\gamma_1^{SS},\gamma_2^{SS},\gamma_3^{SS})$ in the direction of $\tilde{\alpha}^S\times\hat{\alpha}^S$ as well as unit vectors $\hat{\beta}^{SS}=(\hat{\beta}_1^{SS},\hat{\beta}_2^{SS},\hat{\beta}_3^{SS})$ orthogonal to both $\hat{\alpha}^S$ and γ^{SS} and $\tilde{\beta}^{SS}=(\tilde{\beta}_1^{SS},\tilde{\beta}_2^{SS},\tilde{\beta}_3^{SS})$ orthogonal to both $\tilde{\alpha}^S$ and γ^{SS}. We choose the direction of $\hat{\beta}^{SS}$ and $\tilde{\beta}^{SS}$ so that

$$\tilde{\beta}_j^{SS}\hat{\alpha}_j^S=\sin\theta^{SS},\qquad \hat{\beta}_j^{SS}\tilde{\alpha}_j^S=-\sin\theta^{SS}. \tag{2.63-2.64}$$

Amplitudes $\tilde{A}^S_{kl}$ and $\hat{A}^S_{kl}$ of the incoming and outgoing S-waves at the point x in formulas (2.31) and (2.32) can be written in the form

$$\tilde{A}^S_{jp}=\tilde{A}_j^{SS}(\tilde{\beta}_p^{SS}\tilde{a}_1^{SS}+\gamma_p^{SS}\tilde{a}_2^{SS}),\qquad \hat{A}^S_{kl}=\hat{A}_k^{SS}(\hat{\beta}_l^{SS}\hat{a}_1^{SS}+\gamma_l^{SS}\hat{a}_2^{SS}), \tag{2.65-2.66}$$

where $(\hat{a}_1^{SS})^2+(\hat{a}_2^{SS})^2=1$, $(\tilde{a}_1^{SS})^2+(\tilde{a}_2^{SS})^2=1$ and $\tilde{A}_j^{SS}$ and $\hat{A}_k^{SS}$ satisfy the transport equations

$$(\rho^0 c_S^2\tilde{A}_j^{SS}\tilde{A}_j^{SS}\tilde{\phi}^S_{,m})_{,m}=0, \qquad \text{no summation over } j, \tag{2.67}$$

$$(\rho^0 c_S^2\hat{A}_k^{SS}\hat{A}_k^{SS}\hat{\phi}^S_{,m})_{,m}=0, \qquad \text{no summation over } k. \tag{2.68}$$

We also have

$$\tilde{A}^S_{jp}\tilde{\beta}_p^{SS}=\tilde{A}_j^{SS}\tilde{a}_1^{SS},\qquad \tilde{A}^S_{jp}\gamma_p^{SS}=\tilde{A}_j^{SS}\tilde{a}_2^{SS},\qquad \hat{A}^S_{kl}\hat{\beta}_l^{SS}=\hat{A}_k^{SS}\hat{a}_1^{SS},\qquad \hat{A}^S_{kl}\gamma_l^{SS}=\hat{A}_k^{SS}\hat{a}_2^{SS}. \tag{2.69-2.72}$$

Using (2.2), (2.63)-(2.66), (2.69)-(2.72) and (2.45) we obtain

$$\begin{aligned}(\rho'\delta_{pl}&+c'_{lmpq}\tilde{\phi}^S_{,q}\hat{\phi}^S_{,m})(\tilde{\beta}_p^{SS}\tilde{a}_1^{SS}+\gamma_p^{SS}\tilde{a}_2^{SS})(\hat{\beta}_l^{SS}\hat{a}_1^{SS}+\gamma_l^{SS}\hat{a}_2^{SS})\\ &=\rho^0\left(\frac{\rho'}{\rho^0}\cos\theta^{SS}+\frac{\mu'}{\mu^0}\cos 2\theta^{SS}\right)\tilde{a}_1^{SS}\hat{a}_1^{SS}+\rho^0\left(\frac{\rho'}{\rho^0}+\frac{\mu'}{\mu^0}\cos\theta^{SS}\right)\tilde{a}_2^{SS}\hat{a}_2^{SS}.\end{aligned} \tag{2.73}$$

We see that S-waves polarized at the point x parallel and perpendicular to the plane of reflection (the plane containing incoming and outgoing rays at the point x) decouple. Using (2.73) the eq. (2.37) can be rewritten as

$$U_{jk}^{SS}=U_{jk}^{S^1S^1}+U_{jk}^{S^2S^2}, \tag{2.74}$$

where

$$U_{jk}^{S^1S^1}=-\partial_t^2\int_D\rho^0\left[\frac{\rho'}{\rho^0}\cos\theta^{SS}+\frac{\mu'}{\mu^0}\cos 2\theta^{SS}\right]\tilde{A}^S_{jp}\tilde{\beta}_p^{SS}\hat{A}^S_{kl}\hat{\beta}_l^{SS}\delta(t-\tilde{\phi}^S-\hat{\phi}^S)\,\mathrm{d}x, \tag{2.75}$$

$$U_{jk}^{S^2S^2}=-\partial_t^2\int_D\rho^0\left[\frac{\rho'}{\rho^0}+\frac{\mu'}{\mu^0}\cos\theta^{SS}\right]\tilde{A}^S_{jp}\gamma_p^{SS}\hat{A}^S_{kl}\gamma_l^{SS}\delta(t-\tilde{\phi}^S-\hat{\phi}^S)\,\mathrm{d}x. \tag{2.76}$$

AMPLITUDE RADIATION PATTERNS

P-to-P	$\frac{\lambda'}{\lambda^0+2\mu^0}+\frac{\rho'}{\rho^0}\cos\theta^{PP}+\frac{2\mu'}{\lambda^0+2\mu^0}\cos^2\theta^{PP}$
SV-to-SV	$\frac{\rho'}{\rho^0}\cos\theta^{SS}+\frac{\mu'}{\mu^0}\cos 2\theta^{SS}$
SH-to-SH	$\frac{\rho'}{\rho^0}+\frac{\mu'}{\mu^0}\cos\theta^{SS}$
P-to-S	$\frac{\rho'}{\rho^0}\sin\theta^{PS}+\frac{\mu'}{\mu^0}\frac{c_S}{c_P}\sin 2\theta^{PS}$
S-to-P	$-$ (P-to-S)

Fig. 1. Amplitude radiation patterns for elastic scattering. For the acoustic case see formula (1.20).

Like (1.21), eqs. (2.49), (2.56), (2.62), (2.75) and (2.76) are regarded as integral equations for the quantities in square brackets; we gather these amplitude radiation patterns in Fig. 1. Like (1.21), they are generalized Radon transforms with an added angular (θ) dependence. In the next section, following [2, 3] we discuss integral equations of this type and a method for their solution which is implemented in specific cases in Section 4 (acoustics) and Section 5 (elasticity).

3. Inversion of the generalized Radon transform as a canonical problem

Multiparameter inversion in the linearized inverse scattering problems of acoustics and elasticity reduce to inversion of a generalized Radon transform which we will describe now. We consider the transform defined on vector functions $f(x)=(f_1(x), f_2(x), f_3(x))$ by

$$(\mathbf{R}f)(s, r, t)=\sum_{l=1,2,3}\int_D \mathrm{d}x f_l(x)\, w_l(\cos\theta(x, s, r))\, A(x, s, r)\, \delta(t-\tilde{\phi}(s, x)-\hat{\phi}(x, r)). \tag{3.1}$$

where $\tilde{\phi}$ and $\hat{\phi}$ satisfy eikonal equations

$$\sum_l [\tilde{\phi}_{,l}(s, x)]^2=\frac{1}{\tilde{c}^2(x)}, \qquad \sum_l [\hat{\phi}_{,l}(x, r)]^2=\frac{1}{\hat{c}^2(x)}, \tag{3.2-3.3}$$

and

$$\cos\theta(x, s, r)=\tilde{c}(x)\,\hat{c}(x)\,\tilde{\phi}_{,l}(s, x)\,\hat{\phi}_{,l}(x, r). \tag{3.4}$$

Transform (3.1) is a generalization of the transform described in [2, 3], which is motivated by comparing (3.1) with (1.21), (2.49), (2.56), (2.62), (2.75) and (2.76).

Let us denote the total travel time from s to x and then to r by

$$\phi(x, s, r)=\tilde{\phi}(s, x)+\hat{\phi}(x, r), \tag{3.5}$$

and let

$$V(s, r, t)=\partial_t^2(\mathbf{R}f)(s, r, t), \tag{3.6}$$

where $(\mathbf{R}f)$ (s, r, t) is given by (3.1).

Let us apply the following generalized backprojection operators to the function V.

$$(\mathbf{R}_m^* V)(y) = -\frac{1}{\pi^2}\int_{\partial D} \mathrm{d}s \int_{\partial D} \mathrm{d}r \, \frac{B(y,s,r)\, w_m(\cos\theta(y,s,r))\, J(y,s)\, J(y,r)}{A(y,s,r)} \, V(s,r,t)\Big|_{t=\phi(y,s,r)}, \tag{3.7}$$

where $m=1,2,3$ and the functions $J(y,s)$, $J(y,r)$ and $B(y,s,r)$ are yet to be described. We note that in (3.7) the surfaces of sources and of receivers can be different as long as they both are boundaries of domains whose interiors contain the supports of the perturbed parameters. The integration in (3.7) is extended over all sources and receivers.

Let us parametrize the sources and receivers on the surface ∂D by the unit vector tangent to the ray connecting the point y with the source or the receiver. We have $s=s(y,\tilde{\alpha})$, $r=r(y,\hat{\alpha})$ where $\tilde{\alpha}=(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3)$, $\hat{\alpha}=(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3)$, and $\tilde{\alpha}_l=\tilde{c}(x)\tilde{\phi}_{,l}(s,x)$, $\hat{\alpha}_l=\hat{c}(x)\hat{\phi}_{,l}(x,r)$, $l=1,2,3$.

We can now describe functions $J(y,s)$ and $J(y,r)$ in (3.7). These functions are Jacobians of the change of variables from s to $\tilde{\alpha}$ and from r to $\hat{\alpha}$ (see Appendix B, part 2 for the computation of these Jacobians),

$$J(y,s)\,\mathrm{d}s=\mathrm{d}\tilde{\alpha}, \qquad J(y,r)\,\mathrm{d}r=\mathrm{d}\hat{\alpha}. \tag{3.8-3.9}$$

We assume that for all points y and all sources and receivers these Jacobians are positive. For an interpretation of this assumption see [4].

The domains where $\tilde{\alpha}$ and $\hat{\alpha}$ vary are subsets of the unit sphere S^2 which we denote by $\tilde{S}$ and $\hat{S}$. We now rewrite (3.7) in its canonical form

$$(\mathbf{R}_m^* V)(y) = -\frac{1}{\pi^2}\int_{\tilde{S}} \mathrm{d}\tilde{\alpha} \int_{\hat{S}} \mathrm{d}\hat{\alpha} \, \frac{B(y,\tilde{\alpha},\hat{\alpha})\, w_m(\tilde{\alpha}\cdot\hat{\alpha})}{A(y,\tilde{\alpha},\hat{\alpha})} \, V(\tilde{\alpha},\hat{\alpha},t)\Big|_{t=\phi(y,\tilde{\alpha},\hat{\alpha})}, \tag{3.10}$$

where $m=1,2,3$ and $V(\tilde{\alpha},\hat{\alpha},t)=V(s(\tilde{\alpha}),r(\hat{\alpha}),t)$, $A(y,\tilde{\alpha},\hat{\alpha})=A(y,s(\tilde{\alpha}),r(\hat{\alpha}))$, $B(y,\tilde{\alpha},\hat{\alpha})=B(y,s(\tilde{\alpha}),r(\hat{\alpha}))$.

Let $V(s,r,\omega)$ denote the Fourier transform of $V(s,r,t)$ with respect to t. From (3.1) and (3.6) we have

$$V(s,r,\omega)=-\omega^2 \sum_{l=1,2,3} \int_D \mathrm{d}x f_l(x)\, w_l(\cos\theta(x,s,r))\, A(x,s,r)\, \mathrm{e}^{\mathrm{i}\omega\phi(x,s,r)}. \tag{3.11}$$

Using the relation $V(s,r,-\omega)=\bar{V}(s,r,\omega)$, where the bar denotes complex conjugation, the expression in (3.10) can be written as

$$(\mathbf{R}_m^* V)(y) = -\frac{1}{\pi^3}\operatorname{Re}\int_0^\infty \mathrm{d}\omega \int_{\tilde{S}} \mathrm{d}\tilde{\alpha} \int_{\hat{S}} \mathrm{d}\hat{\alpha} \, \frac{B(y,\tilde{\alpha},\hat{\alpha})\, w_m(\tilde{\alpha}\cdot\hat{\alpha})}{A(y,\tilde{\alpha},\hat{\alpha})} \, V(\tilde{\alpha},\hat{\alpha},\omega)\, \mathrm{e}^{-\mathrm{i}\omega\phi(y,\tilde{\alpha},\hat{\alpha})}, \tag{3.12}$$

where $m=1,2,3$.

We substitute (3.11) in (3.12) and consider the resulting expression as a Fourier integral operator $\mathbf{F}$ applied to f,

$$(\mathbf{F}_m f)(y)=\frac{1}{\pi^3}\int_D \mathrm{d}x \int_0^\infty \omega^2\,\mathrm{d}\omega \int_{\tilde{S}} \mathrm{d}\tilde{\alpha} \int_{\hat{S}} \mathrm{d}\hat{\alpha}\, B(y,\tilde{\alpha},\hat{\alpha}) \frac{A(x,\tilde{\alpha},\hat{\alpha})}{A(y,\tilde{\alpha},\hat{\alpha})}$$
$$\cdot\, w_m(\tilde{\alpha}\cdot\hat{\alpha}) \sum_{l=1,2,3} f_l(x)\, w_l(\cos\theta(x,\tilde{\alpha},\hat{\alpha}))\, \mathrm{e}^{\mathrm{i}\omega(\phi(x,\tilde{\alpha},\hat{\alpha})-\phi(y,\tilde{\alpha},\hat{\alpha}))}, \tag{3.13}$$

where $m=1,2,3$.

We have

$$(\mathbf{R}_m^* V)(y)=\operatorname{Re}(\mathbf{F}_m f)(y), \quad m=1,2,3. \tag{3.14}$$

The behavior of the operators $\mathbf{F}_m$ is determined by their phase and amplitude in the neighborhood of the set $C_\Phi = \{(\omega, s, r, x, x) \in R_+ \times \partial D \times \partial D \times D \times D\}$, the projection of which on $D \times D$ is the diagonal. See part 3 of Appendix B for more details. Our next step is to expand the phase of the Fourier integral operator in (3.13) as a Taylor series around the point of reconstruction y. We retain only the first term of the Taylor series to obtain the most singular term of the operator $\mathbf{F}_m$,

$$\phi(x, \tilde{\alpha}, \hat{\alpha}) - \phi(y, \tilde{\alpha}, \hat{\alpha}) \approx (\tilde{\phi}_{,j}(s, y) + \hat{\phi}_{,j}(y, r))(x_j - y_j)$$
$$= \left(\frac{1}{\tilde{c}(y)}\tilde{\alpha}_j + \frac{1}{\hat{c}(y)}\hat{\alpha}_j\right)(x_j - y_j). \tag{3.15}$$

We also expand the weights $A(x, \tilde{\alpha}, \hat{\alpha})$ and $w_l(\cos\theta(x, \tilde{\alpha}, \hat{\alpha}))$ at the point y and keep only the zero order term. Proof that this procedure yields the most singular term follows closely the proof in [2, 3, 4] and is omitted here. On denoting the most singular part of the operator $\mathbf{F}_m$ by $\mathbf{F}_m^0$ we have

$$(\mathbf{F}_m^0 f)(y) = \frac{1}{\pi^3}\int_D \mathrm{d}x \int_0^\infty \omega^2\,\mathrm{d}\omega \int_{\tilde{S}} \mathrm{d}\tilde{\alpha} \int_{\hat{S}} \mathrm{d}\hat{\alpha}\, B(y, \tilde{\alpha}, \hat{\alpha})$$
$$\cdot \sum_{l=1,2,3} f_l(x)\, g_{lm}(\tilde{\alpha}\cdot\hat{\alpha}) \exp\left[\mathrm{i}\omega\left(\frac{1}{\tilde{c}(y)}\tilde{\alpha}_j + \frac{1}{\hat{c}(y)}\hat{\alpha}_j\right)(x_j - y_j)\right], \tag{3.16}$$

where $m = 1, 2, 3$ and

$$g_{lm}(\tilde{\alpha}\cdot\hat{\alpha}) = w_l(\tilde{\alpha}\cdot\hat{\alpha})\, w_m(\tilde{\alpha}\cdot\hat{\alpha}), \quad l, m = 1, 2, 3. \tag{3.17}$$

Further evaluation of (3.16) requires a new system of coordinates at the point y which we introduce below. Let us consider the angle $\theta = \theta(y, s, r)$ defined by $\cos\theta = \tilde{\alpha}_j\hat{\alpha}_j$. The angle θ varies in some subset of $[0, \pi]$ which we denote by E_θ. We also consider the unit vector $\nu = (\nu_1, \nu_2, \nu_3)$ in the direction of the vector $(1/\tilde{c}(y))\tilde{\alpha}_j + (1/\hat{c}(y))\hat{\alpha}_j$,

$$\nu_j = \left(\frac{1}{\tilde{c}(y)}\tilde{\alpha}_j + \frac{1}{\hat{c}(y)}\hat{\alpha}_j\right) \Big/ \left|\frac{1}{\tilde{c}(y)}\tilde{\alpha} + \frac{1}{\hat{c}(y)}\hat{\alpha}\right|, \quad j = 1, 2, 3, \tag{3.18}$$

and

$$\left|\frac{1}{\tilde{c}(y)}\tilde{\alpha} + \frac{1}{\hat{c}(y)}\hat{\alpha}\right| = \left(\frac{\Delta^2(y) + 4\cos^2\frac{1}{2}\theta}{\tilde{c}(y)\,\hat{c}(y)}\right)^{1/2}. \tag{3.19}$$

Here

$$\Delta(y) = |\tilde{c}(y) - \hat{c}(y)|/(\tilde{c}(y)\,\hat{c}(y))^{1/2}, \tag{3.20}$$

is a nonnegative dimensionless parameter. If $\tilde{c}(y) = \hat{c}(y)$ then $\Delta(y) = 0$.

The unit vector ν varies over $S_\nu = S_\nu(\theta)$, a subset of the unit sphere S^2. This subset, in general, depends on the angle θ. Given ν and θ we can find vectors $\tilde{\alpha}$ and $\hat{\alpha}$ up to the orientation of the plane which contains these vectors. To define the orientation of this plane we introduce an additional angle $\psi = \psi(y, \nu)$. This is an angle in the plane orthogonal to the unit vector ν and determines the orientation of the plane containing vectors $\tilde{\alpha}$ and $\hat{\alpha}$. If sources and receivers are available at all points on the boundary ∂D then $0 \leq \psi < 2\pi$. Otherwise, ψ will vary in some (possibly disconnected) subset of $[0, 2\pi]$ which we denote by $E_\psi = E_\psi(\nu, \theta)$.

We choose the weight $B(y, \tilde{\alpha}, \hat{\alpha})$ in (3.16) to be of the form

$$B(y, \tilde{\alpha}, \hat{\alpha}) = B(y, \tilde{\alpha} \cdot \hat{\alpha}) = \frac{1}{16 \operatorname{mes} E_\psi} \frac{1}{\cos \frac{1}{2}\theta} \left(\frac{\Delta^2 + 4\cos^2 \frac{1}{2}\theta}{\tilde{c}(y)\, \hat{c}(y)} \right)^{3/2} b(y, \theta), \tag{3.21}$$

where $\operatorname{mes} E_\psi = \int_{E_\psi(\nu,\theta)} \mathrm{d}\psi$, and $b(y, \theta)$ is a function yet to be described.

Using the relation (Burridge and Beylkin, [30])

$$\mathrm{d}\tilde{\alpha}\, \mathrm{d}\hat{\alpha} = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\nu\, \mathrm{d}\psi, \tag{3.22}$$

(see Appendix C) and integrating over E_ψ we arrive at

$$(\mathbf{F}^0_m f)(y) = \frac{1}{(2\pi)^3} \int_D \mathrm{d}x \int_0^\infty \omega^2\, \mathrm{d}\omega \int_{E_\theta} \mathrm{d}\theta \int_{S_\nu} \mathrm{d}\nu\, \sin \tfrac{1}{2}\theta \left(\frac{\Delta^2 + 4\cos^2 \frac{1}{2}\theta}{\tilde{c}(y)\hat{c}(y)} \right)^{3/2}$$
$$\cdot\, b(y, \theta) \sum_{l=1,2,3} f_l(x)\, g_{lm}(\cos\theta) \exp\left[\mathrm{i}\omega \left| \frac{1}{\tilde{c}(y)} \tilde{\alpha} + \frac{1}{\hat{c}(y)} \hat{\alpha} \right| \nu_j (x_j - y_j) \right], \tag{3.23}$$

where $m = 1, 2, 3$.

We proceed to evaluate the expression in (3.23) with limits of integration with respect to ω replaced by $\omega_{\min}$ and $\omega_{\max}$. This allows us to account for the case when the useful spectrum of the signal is bandlimited within the interval $(\omega_{\min}, \omega_{\max})$. We change the variable ω in the expression (3.23) to ω' given by

$$\omega' = \omega \left| \frac{1}{\tilde{c}(y)} \tilde{\alpha} + \frac{1}{\hat{c}(y)} \hat{\alpha} \right|, \tag{3.24}$$

and arrive at

$$(\mathbf{F}^0_m f)(y) = \frac{1}{(2\pi)^3} \int_{E_\theta} \mathrm{d}\theta \int_{\omega'_{\min}}^{\omega'_{\max}} \omega'^2\, \mathrm{d}\omega' \int_{S_\nu} \mathrm{d}\nu \int_D \mathrm{d}x\, \sin \tfrac{1}{2}\theta$$
$$\cdot\, b(y, \theta) \sum_{l=1,2,3} f_l(x)\, g_{lm}(\cos\theta)\, \mathrm{e}^{\mathrm{i}\omega' \nu_j (x_j - y_j)}, \quad m = 1, 2, 3, \tag{3.25}$$

where

$$\omega'_{\min} = \left(\frac{\Delta^2(y) + 4\cos^2 \frac{1}{2}\theta}{\tilde{c}(y)\, \hat{c}(y)} \right)^{1/2} \omega_{\min}, \qquad \omega'_{\max} = \left(\frac{\Delta^2(y) + 4\cos^2 \frac{1}{2}\theta}{\tilde{c}(y)\, \hat{c}(y)} \right)^{1/2} \omega_{\max}. \tag{3.26-3.27}$$

We introduce $\langle f_l(y) \rangle$, where

$$\langle f_l(y) \rangle = \frac{1}{(2\pi)^3} \int_{\Omega_\theta} \omega'^2\, \mathrm{d}\omega'\, \mathrm{d}\nu \int_D \mathrm{d}x f_l(x)\, \mathrm{e}^{\mathrm{i}\omega' \nu_j (x_j - y_j)}, \quad l = 1, 2, 3. \tag{3.28}$$

The inverse spatial Fourier transform in (3.28) is over the domain

$$\Omega_\theta = (\omega'_{\min}, \omega'_{\max}) \times S_\nu. \tag{3.29}$$

Using the notion of the wave front set (see Appendix D) it is clear that (3.28) can be considered a good approximation to $f_l(y)$ if WF $(f_l(y))$ contains only directions which are covered by $D \times \Omega_\theta$. If this is the case then, according to the definition of the wave front set, the Fourier transform decays rapidly in directions which are not covered by $D \times \Omega_\theta$ and one can expect a reasonable approximation through (3.28). In case a discontinuity occurs on a smooth surface this condition simply means that we can obtain a good bandlimited reconstruction only if we observe specular reflections from the surface.

Introducing new variables in (3.25) $p_j = \omega' \nu_j$, where $j = 1, 2, 3$, into (3.28), we have

$$(\mathbf{F}_m^0 f)(y) = \int_{E_\theta} \mathrm{d}\theta \sin \tfrac{1}{2}\theta \, b(y, \theta) \sum_{l=1,2,3} \langle f_l(y) \rangle \, g_{lm}(\cos \theta), \quad m = 1, 2, 3. \tag{3.30}$$

Let us assume that Ω_θ does not depend on θ. This is true, for example, if $\omega_{\min} = 0$ and $\omega_{\max} = \infty$ and there are enough sources and receivers to maintain the same aperture for all angles θ involved. In practice independence of Ω_θ on θ can be arranged in a variety of ways, provided there is sufficient coverage and the frequency band $(\omega_{\min}, \omega_{\max})$ is sufficiently wide.

We have

$$(\mathbf{F}_m^0 f)(y) = \sum_{l=1,2,3} \langle f_l(y) \rangle \int_{E_\theta} \mathrm{d}\theta \sin \tfrac{1}{2}\theta \, b(y, \theta) \, g_{lm}(\cos \theta); \quad m = 1, 2, 3. \tag{3.31}$$

It follows from (3.14) and (3.31) that

$$(\mathbf{R}_m^* U)(y) = (\mathbf{F}_m^0 f)(y) + \mathrm{Re}(\mathbf{T}_m f)(y), \quad m = 1, 2, 3, \tag{3.32}$$

where the operator $\mathbf{T}_m = \mathbf{F}_m - \mathbf{F}_m^0$. The operator $\mathbf{T}_m$ can be shown to be at least of the class $L^{-1}(D)$ as a pseudodifferential operator. Roughly this means that the operator produces a function with one more derivative than the function to which it is applied. Defining

$$f_m^{\mathrm{est}}(y) = (\mathbf{R}_m^* V)(y), \tag{3.33}$$

and neglecting smoother terms we have

$$f_m^{\mathrm{est}}(y) \approx (\mathbf{F}_m^0 f)(y). \tag{3.34}$$

Thus, we arrive at the linear system

$$f_m^{\mathrm{est}}(y) = \sum_{l=1,2,3} \langle f_l(y) \rangle \, a_{lm}(y), \quad m = 1, 2, 3, \tag{3.35}$$

where

$$a_{lm}(y) = \int_{E_\theta} b(y, \theta) \sin \tfrac{1}{2}\theta \, g_{lm}(\cos \theta) \, \mathrm{d}\theta, \quad l, m = 1, 2, 3, \tag{3.36}$$

and g_{lm} are given by (3.17).

Solving the linear system (3.35) yields the functions $\langle f_l \rangle$ that we are seeking.

Remark 1. If $\tilde{c} \neq \hat{c}$ and $\omega_{\min} \neq 0$, then we note that $\omega'_{\min}$ cannot be zero for any angle, since $\omega'_{\min} > \omega_{\min} \Delta / (\tilde{c}\hat{c})^{1/2}$. This produces a 'hole' near zero in the domain of spatial frequences Ω_θ and, thus, excluding the possibility of recovering very low spatial frequences in this case. Condition $\tilde{c} \neq \hat{c}$ is met when we consider P-to-S or S-to-P converted waves.

Remark 2. If we multiply $V(s, r, \omega)$ in (3.11) by $(\mathrm{i}\omega)^d$ and apply the generalized backprojection operator to $(\mathrm{i}\omega)^d V(s, r, \omega)$, then we need to change the weight $B(y, \tilde{\alpha}, \hat{\alpha})$ in (3.21) as follows

$$B(y, \tilde{\alpha}, \hat{\alpha}) = \frac{1}{16 \,\mathrm{mes}\, E_\psi} \frac{1}{\cos \frac{1}{2}\theta} \left(\frac{\Delta^2 + 4\cos^2 \frac{1}{2}\theta}{\tilde{c}(y)\, \hat{c}(y)} \right)^{(3+d)/2} b(y, \theta). \tag{3.37}$$

In this case instead of $\langle f_l(y) \rangle$ in (3.28) we reconstruct

$$\langle \tilde{f}_l(y) \rangle = \frac{1}{(2\pi)^3} \int_{\Omega_\theta} \mathrm{d}p \, |p|^d \int_D \mathrm{d}x \, f_l(x) \, \mathrm{e}^{\mathrm{i}p_j(x_j - y_j)}, \tag{3.38}$$

where $l = 1, 2, 3$.

4. Asymptotic solution of linearized inverse problem for acoustics

To see the connection of the linearised inverse problem for acoustics with the inversion of the generalized Radon transform as defined in (3.1) we set

$$f = (\kappa'/\kappa^0, \sigma'/\sigma^0, 0), \tag{4.1}$$

$$w_l(\cos\theta) = \cos^{l-1}\theta, \quad l = 1, 2, \tag{4.2}$$

$$\tilde{c}(x) = \hat{c}(x) = c^0(x), \tag{4.3}$$

$$A(x, s, r) = \kappa^0(x)\, \tilde{A}(s, x)\, \hat{A}(x, r). \tag{4.4}$$

Then, from (1.21) and (3.1), we obtain

$$U(s, r, t) = -\partial_t^2(\mathbf{R}f)(s, r, t), \tag{4.5}$$

where we may eliminate f_3 from our notation and consider $f = (f_1, f_2)$. Relation (4.5) reduces the linearized inverse scattering problem of acoustics to the inversion of the transform (3.1). We can now use the results of the previous section to obtain suitable inversion formulas.

According to (3.7), (3.33) and (4.2)-(4.4), for each point y and $m = 1, 2$ we compute

$$f_m^{\text{est}}(y) = \frac{1}{\pi^2} \int_{\partial D} \mathrm{d}s \int_{\partial D} \mathrm{d}r \frac{B(y, s, r)\, J(y, s)\, J(y, r)}{\kappa^0(y)\, \tilde{A}(s, y)\, \hat{A}(y, r)} \cos^{m-1}\theta(y, s, r)\, U(s, r, t)|_{t=\phi(y,s,r)}, \tag{4.6}$$

where

$$B(y, s, r) = \frac{b(y, \theta) \cos^2 \frac{1}{2}\theta}{2 \operatorname{mes} E_\psi\, [c^0(y)]^3}. \tag{4.7}$$

We also compute

$$a_{lm}(y) = \int_{E_\theta} b(y, \theta) \sin \tfrac{1}{2}\theta \cos^{l+m-2}\theta\, \mathrm{d}\theta, \quad l, m = 1, 2. \tag{4.8}$$

We then solve the linear system (3.35) (which in this case is a 2×2 system) to find f_1 and f_2, which according to (4.1) are κ'/κ^0 and σ'/σ^0.

We have applied our algorithm to synthetic acoustic data generated by a finite difference code developed at the University of Wyoming. The finite difference program had to be executed for each of the eighty source positions and we wish to thank Walter R. Fletcher of the University of Wyoming for creating this extensive data set. We used the scattered field at eighty-three receiver locations for each source location along the surface. A typical shot gather is shown in Fig. 2. We note that the traces in this data set show evidence of considerable numerical dispersion. Since we use the amplitude information, this may have adversely affected some of the results. Also, the sources were not precisely on the boundary but slightly below. Thus, the direct wave results from interference between radiation from the source and from its image. The same is also true for the receivers.

The model is described in diagrams in Figs. 3 and 4 and consists of eleven homogeneous regions which can be thought of as five layers to the left and six to the right of a highly deformed fault. Owing to the

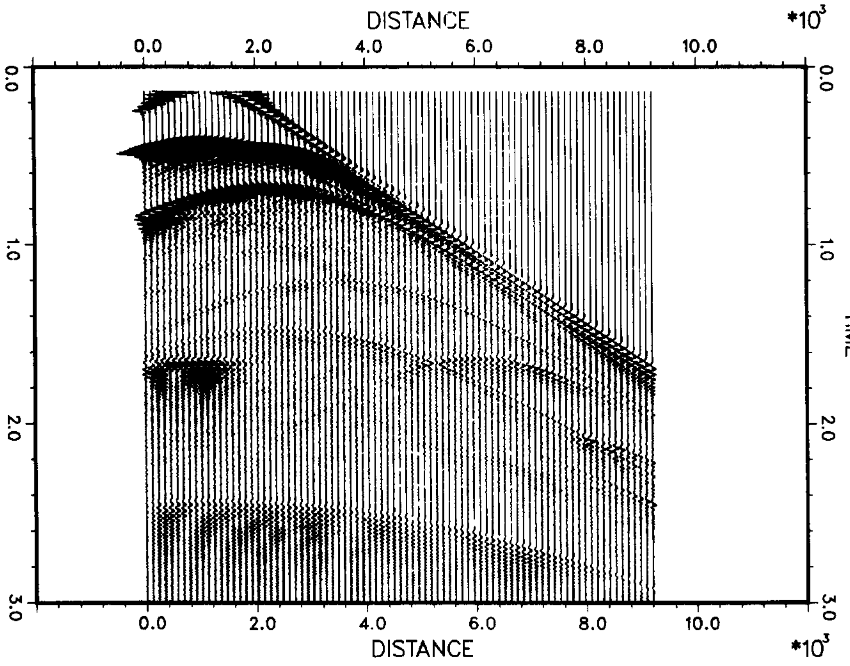

Fig. 2. A typical source gather.

extreme deformation, the reflecting interfaces separating the homogeneous regions present a wide range of directions and thus form a rather severe test for our algorithm. On the other hand, we kept the deviation of the velocity at about 2% to 4% and deviations of σ and ρ less than about 8% so as to remain within the range of the single scattering approximation. Figs. 3 and 4 show the perturbations σ'/σ_0 and κ'/κ_0 for the model. The background medium was chosen to be homogeneous.

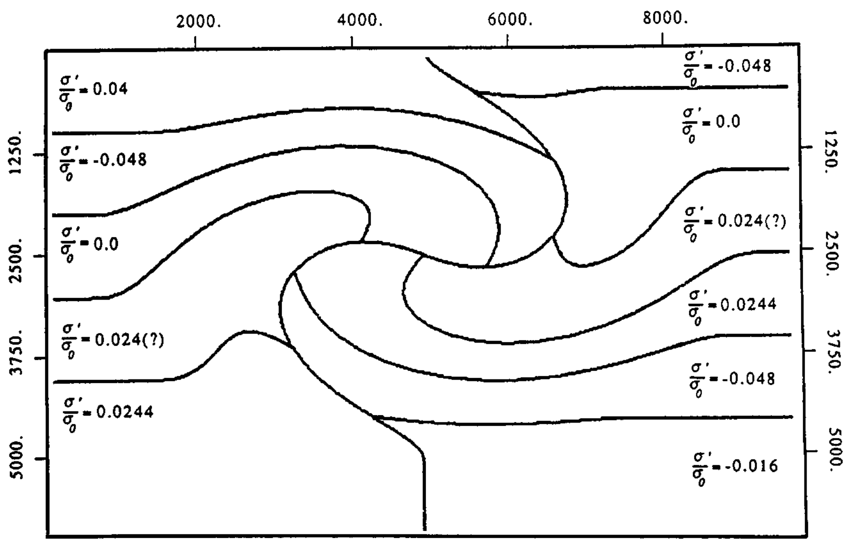

Fig. 3. The specific volume model showing relative perturbation with respect to the background medium.

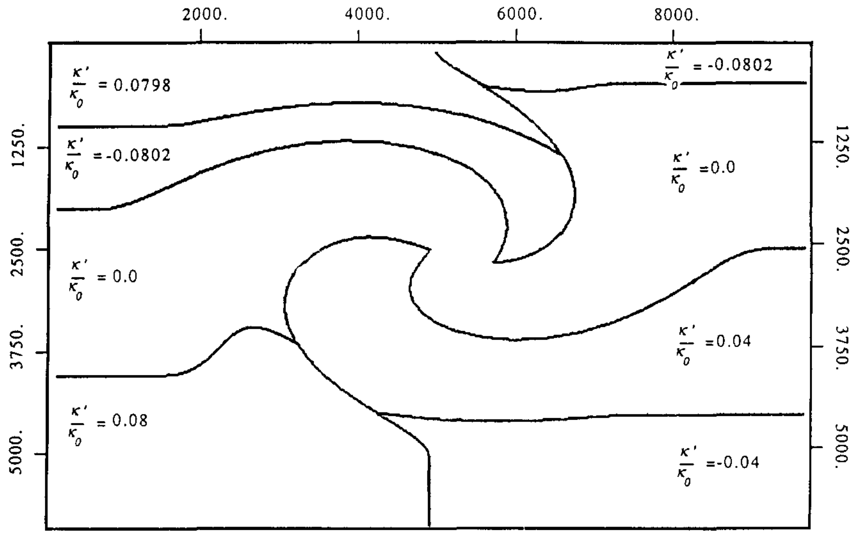

Fig. 4. The compressibility model showing relative perturbation with respect to the background medium.

The algorithm in its two-dimensional form was applied and the results are shown in Figs. 5 and 6. Because of numerical dispersion and the shape of the incident wavelet, we obtain only an approximation of the quantities σ'/σ_0 and κ'/κ_0 rather than these quantities themselves. Thus, at a step discontinuity of perturbed specific volume and compressibility we see plotted not a step discontinuity but a smoothed

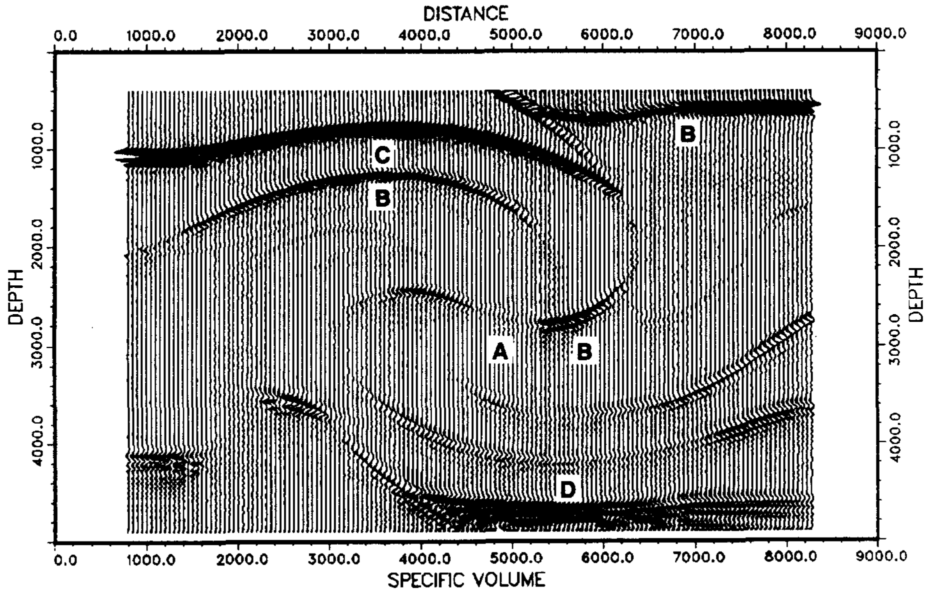

Fig. 5. Reconstruction of the specific volume.

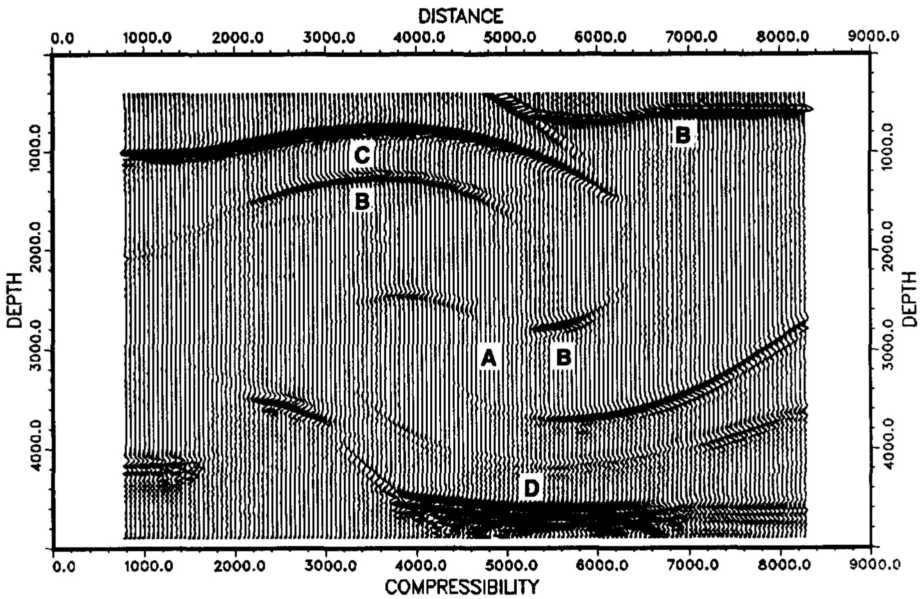

Fig. 6. Reconstruction of the compressibility.

derivative of the delta function of (signed) distance from the interface with amplitude ideally proportional to the jump in the original quantities σ'/σ_0 and κ'/κ_0.

For the method to work best, at each image point the unit vector ν (the bisector of the angle between incident and received ray) should range over an angular sector of π. However, because of the limited range of positions of sources and receivers (limited aperture), the range of ν is limited and we observe several effects of this. If the scattering occurs so that no source-receiver pair is able to 'see' a specular reflection our algorithm produces almost zero image. Incomplete coverage also degrades the image. However, as our numerical simulations indicate, these effects do not strongly affect the ability to separate the parameters, though the size of the jump recovered by the algorithm is less reliable. We feel that further effort is required to alleviate these limited aperture effects.

In the present example, in spite of the imperfections mentioned above (and similar imperfections are always present in real field data), the relative amplitudes of the jump in σ'/σ_0 and κ'/κ_0 at similar locations are accurate. Thus, we see that in the center of the diagram in Fig. 4 at location A, where κ'/κ_0 is continuous across the short segment of the fault, that segment does not appear in the κ'/κ_0 image in Fig. 6. On the other hand across the same segment σ'/σ_0 suffers a jump (Fig. 3) and this segment does appear in the σ'/σ_0 image (Fig. 5). The amplitudes at the three places marked B are about the same and indeed the jumps in σ'/σ_0 and κ'/κ_0 are the same at these interfaces. The jump at C is about twice that at B and with the opposite sign. This is also borne out in the images. The jumps at D are comparable in magnitude with those at B. Again this is roughly borne out in the images, but the considerable numerical dispersion, which is stronger at greater depths, smears the image at D. Analyzing the image, we have enough confidence to conclude that there must have been an error in the input of the data in the forward model of σ. We have marked by question mark in Fig. 3 the data about which we have doubts.

Our numerical simulations clearly show that the parameters can be separately estimated from the data.

5. Asymptotic inversion of individual elastic scattering modes

The inversion problem in the elastic medium is more complicated than that in the acoustic medium due to multiple modes of scattering, namely, P-to-P, P-to-S, S-to-P, S-to-S scattering. On the other hand multiple modes of scattering provide additional information. We consider inversion of individual scattering modes, but inversion of several scattering modes can easily be combined into a single procedure.

P-to-P scattering. Using the notations of Section 2 and writing $U_{jk} = U_{jk}^{PP}$, $\theta = \theta^{PP}$ we set

$$f = \left(\frac{\lambda'}{\lambda^0 + 2\mu^0}, \frac{\rho'}{\rho^0}, \frac{2\mu'}{\lambda^0 + 2\mu^0} \right), \tag{5.1}$$

$$w_l(\cos\theta) = \cos^{l-1}\theta, \quad l = 1, 2, 3, \tag{5.2}$$

$$\tilde{c}(x) = \hat{c}(x) = c_P(x), \tag{5.3}$$

$$A(x, s, r) = A_{jk}(x, s, r) = \rho^0(x)\, \tilde{A}_j^P(s, x)\, \hat{A}_k^P(x, r) \tag{5.4}$$

and obtain from (2.49) and (3.1)

$$U_{jk}(s, r, t) = -\partial_t^2(\mathbf{R}f)(s, r, t). \tag{5.5}$$

In principle, each component U_{jk} can be inverted separately. In practice, seismic sources usually excite more than one component. In this case what is measured is a linear combination of the fields generated by single component sources. Since only the amplitude terms are affected, what follows can be easily adjusted for that case.

In what follows we invert all components simultaneously. This requires the ordinary projection of the field on the direction of the amplitude vector computed within the background medium. The derivation follows step by step the one presented in Section 3, as the reader may verify.

According to (3.7), (3.33) and (5.2)-(5.4) for each point y and $m = 1, 2, 3$ we compute

$$f_m^{\text{est}}(y) = \frac{1}{\pi^2 \rho^0(y)} \int_{\partial D} \mathrm{d}s \int_{\partial D} \mathrm{d}r\, B(y, s, r)\, J(y, s)\, J(y, r)$$
$$\cdot \cos^{m-1}\theta(y, s, r) \frac{\tilde{A}_j^P(s, y)\, \hat{A}_k^P(y, r)}{\|\tilde{A}^P(s, y)\|^2 \|\hat{A}^P(y, r)\|^2} U_{jk}(s, r, t) \Big|_{t = \phi(y, s, r)}, \tag{5.6}$$

where $\tilde{A}_j^P$, $\hat{A}_k^P$ are described in (2.46) and (2.47), $J(y, s)$, $J(y, r)$ are Jacobians described in (3.8), (3.9) and

$$\|\tilde{A}^P\|^2 = \sum_{j=1,2,3} (\tilde{A}_j^P)^2, \qquad \|\hat{A}^P\|^2 = \sum_{k=1,2,3} (\hat{A}_k^P)^2. \tag{5.7-5.8}$$

According to (3.21) and (5.3)

$$B(y, s, r) = \frac{b(y, \theta) \cos^2 \frac{1}{2}\theta}{2 \operatorname{mes} E_\psi\, [c_P(y)]^3}. \tag{5.9}$$

We also compute

$$a_{lm}(y) = \int_{E_\theta} b(y, \theta) \sin \tfrac{1}{2}\theta \cos^{l+m-2}\theta\, \mathrm{d}\theta, \quad l, m = 1, 2, 3. \tag{5.10}$$

We reconstruct the perturbations of the three elastic parameters in (5.1) by solving the 3×3 linear system (3.35) with a_{lm} given by (5.10) and f_m^{est} by (5.6).

P-to-S scattering. Using the notations of Section 2 and writing $U_{jk} = U_{jk}^{PS}$, $\theta = \theta^{PS}$ we set

$$f = \left(\frac{\rho'}{\rho^0}, \frac{\mu'}{\mu^0}\frac{c_S}{c_P}, 0\right), \tag{5.11}$$

$$w_l(\cos\theta) = \sin\theta\,(2\cos\theta)^{l-1}, \quad l = 1, 2, \tag{5.12}$$

$$\tilde{c}(x) = c_P(x), \qquad \hat{c}(x) = c_S(x), \tag{5.13-5.14}$$

$$A(x, s, r) = A_{jk}(x, s, r) = \rho^0(x)\,\tilde{A}_j^P(s, x)\,\hat{A}_{kl'}^S(y, r)\,\hat{\beta}_{l'}^{PS}. \tag{5.15}$$

We compute

$$f_m^{\text{est}}(y) = \frac{1}{\pi^2\rho^0(y)}\int_{\partial D} \mathrm{d}s \int_{\partial D} \mathrm{d}r\, B(y, s, r)\, J(y, s)\, J(y, r)$$

$$\cdot \sin\theta\cos^{m-1}\theta\frac{\tilde{A}_j^P(s, y)\,\hat{A}_{kl'}^S(y, r)\,\hat{\beta}_{l'}^{PS}}{\|\tilde{A}^P(s, y)\|^2\|\hat{A}^{PS}(y, r)\|^2}\, U_{jk}(s, r, t)\bigg|_{t=\phi(y,s,r)}, \quad m = 1, 2, \tag{5.16}$$

where $\|\tilde{A}^P\|^2$ is defined in (5.7) and

$$\|\hat{A}^{PS}\|^2 = \sum_{k=1,2,3} (\hat{A}_{kl'}^S\hat{\beta}_{l'}^{PS})^2. \tag{5.17}$$

According to (3.21), (5.13) and (5.14) we have

$$B(y, s, r) = \frac{1}{16\,\text{mes}\,E_\psi}\frac{1}{\cos\frac{1}{2}\theta}\left(\frac{\Delta^2(y) + 4\cos^2\frac{1}{2}\theta}{c_P(y)\,c_S(y)}\right)^{3/2} b(y, \theta), \tag{5.18}$$

where Δ is given by (3.20) in combination with (5.13) and (5.14).

We also compute

$$a_{lm}(y) = \int_{E_\theta} b(y, \theta)\sin\tfrac{1}{2}\theta\sin^2\theta\,(2\cos\theta)^{l+m-2}\,\mathrm{d}\theta, \quad l, m = 1, 2. \tag{5.19}$$

This time we solve the system (3.35) with a_{lm} given by (5.19) and f_m^{est} by (5.16).

S-to-P scattering. This case is completely analogous to the case of *P*-to-*S* scattering as can be seen by comparing (2.56) and (2.62). The only difference in the construction of the generalized backprojection operator as compared with (5.16) is the sign and exchange of roles between *P*- and *S*-amplitudes.

S-to-S scattering. Due to the polarization of the *S*-waves there are two subcases as is apparent from (2.75) and (2.76). Given the point of reconstruction and the position of source and receiver the orthogonal pairs of unit vectors γ^{SS} and $\tilde{\beta}^{SS}$ as well as γ^{SS} and $\hat{\beta}^{SS}$ (for definition of these vectors see Section 2, *S*-to-*S* scattering) are used to obtain projections of the amplitudes of *S*-waves. Thus, we get the separation of the *S*-to-*S* scattered field into two components according to (2.74). In the case of a constant background the two components S^1 and S^2 can be identified with *SV*- and *SH*-waves. These two components can be inverted separately.

For S^1-to-S^1 scattering using notation $\theta = \theta^{SS}$ we set

$$f = (\rho'/\rho^0, \mu'/\mu^0), \tag{5.20}$$

$$w_l(\cos\theta) = \cos^{2-l}\theta\cos^{l-1}2\theta, \quad l = 1, 2, \tag{5.21}$$

$$\tilde{c}(x) = \hat{c}(x) = c_S(x), \tag{5.22}$$

$$A(x, s, r) = A_{jk}(x, s, r) = \rho^0(x)\,\tilde{A}_{jp}^S(s, x)\,\tilde{\beta}_p^{SS}\,\hat{A}_{kl}^S(x, r)\,\hat{\beta}_l^{SS}. \tag{5.23}$$

We compute for $m = 1, 2$

$$f_m^{\text{est}}(y) = \frac{1}{\pi^2 \rho^0(y)} \int_{\partial D} \mathrm{d}s \int_{\partial D} \mathrm{d}r\, B(y, s, r)\, J(y, s)\, J(y, r) \cdot \cos^{2-m}\theta \cos^{m-1} 2\theta \frac{\tilde{A}_{jp}^{SS}(s, y)\, \tilde{\beta}_p^{SS}\, \hat{A}_{kl}^{S}(y, r)\, \hat{\beta}_l^{SS}}{\|\tilde{A}^{SS}(s, y)\|_1^2\, \|\hat{A}^{SS}(y, r)\|_1^2} U_{jk}^{S^1S^1}(s, r, t)\Big|_{t=\phi(y,s,r)}, \tag{5.24}$$

where

$$\|\hat{A}^{SS}\|_1^2 = \sum_{k=1,2,3} (\hat{A}_{kl}^{S}\hat{\beta}_l^{SS})^2, \qquad \|\tilde{A}^{SS}\|_1^2 = \sum_{j=1,2,3} (\tilde{A}_{jp}^{S}\tilde{\beta}_p^{SS})^2, \tag{5.25–5.26}$$

and

$$B(y, s, r) = \frac{b(y, \theta)\cos^2\frac{1}{2}\theta}{2 \operatorname{mes} E_\psi\, [c_S(y)]^3}. \tag{5.27}$$

We also compute

$$a_{lm}(y) = \int_{E_\theta} b(y, \theta) \sin\tfrac{1}{2}\theta \cos^{l+m-2} 2\theta \cos^{4-l-m}\theta\, \mathrm{d}\theta, \quad l, m = 1, 2. \tag{5.28}$$

Solving the system (3.35) with a_{lm} given by (5.28) and f_m^{est} by (5.24) yields the perturbations of the parameters in (5.20) which we are seeking.

For S^2-to-S^2 scattering we set

$$f = (\rho'/\rho^0, \mu'/\mu^0), \tag{5.29}$$

$$w_l(\cos\theta) = \cos^{l-1}\theta, \quad l = 1, 2, \tag{5.30}$$

$$\tilde{c}(x) = \hat{c}(x) = c_S(x), \tag{5.31}$$

$$A(x, s, r) = A_{jk}(x, s, r) = \rho^0(x)\, \tilde{A}_{jp}^{S}(s, x)\, \gamma_p^{SS}\, \hat{A}_{kl}^{S}(x, r)\, \gamma_l^{SS}. \tag{5.32}$$

We compute

$$f_m^{\text{est}}(y) = \frac{1}{\pi^2 \rho^0(y)} \int_{\partial D} \mathrm{d}s \int_{\partial D} \mathrm{d}r\, B(y, s, r)\, J(y, s)\, J(y, r) \cdot \cos^{m-1}\theta \frac{\tilde{A}_{jp}^{SS}(s, y)\, \gamma_p^{SS}\, \hat{A}_{kl}^{S}(y, r)\, \gamma_l^{SS}}{\|\tilde{A}^{SS}(s, y)\|_2^2\, \|\hat{A}^{SS}(y, r)\|_2^2} U_{jk}^{S^2S^2}(s, r, t)\Big|_{t=\phi(y,s,r)}, \quad m = 1, 2, \tag{5.33}$$

where B is given by (5.27) and

$$\|\hat{A}^{SS}\|_2^2 = \sum_{k=1,2,3} (\hat{A}_{kl}^{S}\gamma_l^{SS})^2, \qquad \|\tilde{A}^{SS}\|_2^2 = \sum_{j=1,2,3} (\tilde{A}_{jp}^{S}\gamma_p^{SS})^2. \tag{5.34–5.35}$$

We also compute

$$a_{lm}(y) = \int_{E_\theta} b(y, \theta) \sin\tfrac{1}{2}\theta \cos^{l+m-2}\theta\, \mathrm{d}\theta. \tag{5.36}$$

Solving the system (3.35) with a_{lm} given by (5.36) and f_m^{est} by (5.33) yields the perturbations of the parameters in (5.29).

6. The Kirchhoff approximation

It is often reasonable to assume that the discontinuities in the parameters are located on smooth surfaces. While the Born approximation is useful, the Kirchhoff approximation might then be a better alternative. Bleistein [33] has proposed a modification of the algorithms described in [4–6] using the Kirchhoff approximation in the forward problem instead of the Born. Applying the method of stationary phase, he justified the modification of the inversion operator for the case when the location of source or receiver is a function of the other, as for example in a common midpoint gather or a shot gather. He also proposed to perform inversion twice and then recover the angle at the specular point from the ratio of magnitudes of the two images (at selected points). This results in a reconstruction of the reflection coefficient as a function of angle. Parsons [34] has used this scheme with the reflection coefficient linearized.

While such a scheme is valid and useful it has several weakenesses. First, at a certain point of the procedure one has to find the maximum value within a spatial window in the reconstructed image, and due to noise this might be a difficult task for real data. Second, the angle at the specular point relies on amplitudes which are less accurately determined than the phases. So, in applying the method, one would prefer to obtain the angle at the specular point independently (one possibility would be to find the local dip from a prestack-migration image). Third, when inverting for parameters such a scheme would necessitate storing an intermediate image for each of many angles because the reflection coefficient is a function not only of position but of the angle as well.

In this section on the example of acoustic and P-to-P elastic scattering we will show that the inversion scheme described in Section 3 allows us to use the Kirchhoff approximation in the forward problem. The generalized backprojection operator is the same as for the Born approximation. We show that the difference in images of the parameters obtained under the assumption of the Kirchhoff approximation compared with those for the Born approximation is equivalent to the effect of a spatial filter. Thus, one need not decide which approximation in the forward model is better for a given situation until examining the images after the reconstruction. We note that our scheme does not require explicit recovery of the angle at the specular point.

First some preliminary remarks. The singly scattered field in the Kirchhoff approximation can be written in the frequency domain as

$$U(s, r, \omega) = \mathrm{i}\omega \int_\Gamma \mathrm{d}\Gamma\, R(x, \theta_s)\, \tilde{A}(s, x)\, \hat{A}(x, r)\, \mathrm{N}_x \cdot \nabla\phi(x, s, r)\, \mathrm{e}^{\mathrm{i}\omega\phi(x,s,r)}, \tag{6.1}$$

where ϕ denotes the total travel time from s to x on the surface Γ and then to r:

$$\phi(x, s, r) = \tilde{\phi}(s, x) + \hat{\phi}(x, r), \tag{6.2}$$

and N_x is the outer normal to Γ.

When the change in parameters across the interface Γ is small, it is easy to verify by comparison with Aki and Richards [38, p. 153] that the reflection coefficient $R(x, \theta_s)$ in (6.1) is related to the amplitude radiation patterns $f(x, \theta)$ derived using the Born approximation. Let $f(x, \theta)$ be given by (1.20) for acoustics and by the expression in square brackets in (2.49) for elasticity (see Fig. 1). Then, for the corresponding reflection coefficients, we have

$$R(x, \theta_s) = f(x, 2\theta_s)/4\cos^2\theta_s, \tag{6.3}$$

where θ_s is the angle between the normal and the ray connecting the source with the point x. At the specular point (6.3) becomes

$$R(x, \tfrac{1}{2}\theta) = f(x, \theta)/4\cos^2\tfrac{1}{2}\theta, \tag{6.4}$$

where θ is the angle between the rays connecting source and receiver to the point x.

We also observe that at the specular point

$$\mathrm{N}_x \cdot \nabla\phi(x, s, r) = |\nabla\phi(x, s, r)| = (2\cos\tfrac{1}{2}\theta)/c(x), \tag{6.5}$$

where $c(x)$ is the velocity.

We now rewrite (6.1) as a volume integral using the singular function $\gamma(x)$ of the surface:

$$U(s, r, \omega) = \mathrm{i}\omega \int_D \mathrm{d}x\, \gamma(x)\, R(x, \theta_s)\, \tilde{A}(s, x)\, \hat{A}(x, r)\, \mathrm{N}_x \cdot \nabla\phi(x, s, r)\, \mathrm{e}^{\mathrm{i}\omega\phi(x,s,r)}, \tag{6.6}$$

and compare it with (1.21) and (2.49) rewritten in the frequency domain. In view of (6.4) and (6.5) one can see that an extra factor $\mathrm{i}\omega$ is present in (1.21) whereas an extra angular dependent factor is present in (6.6). We will show that these are the only differences that should be taken into account in the inversion formulas. Bleistein's modification [33] amounts precisely to such a change in the inversion operator and he verified this using the method of stationary phase when the locations of sources and receivers are functions of one parameter.

For multiple sources and receivers we outline a proof, based on the theory of pseudodifferential operators, and describe it by comparison with the derivation in Section 3 so that only the differences need to be described in detail. To justify our derivation we use the so-called microlocalization technique which will allow us to set the angle in the amplitude of a pseudodifferential operator to be that at a specular point and will guarantee that the dropped terms are smooth.

Assuming small changes in parameters across the interface Γ we write (6.3) as

$$R(x, \theta_s) = \frac{1}{4\cos^2\theta_s} \sum_{l=1,2,3} f_l(x)\, w_l(\cos 2\theta_s(x, s, r)), \tag{6.7}$$

where the angle-dependent coefficients w_l are the same as for the Born approximation.

We replace (3.11) by $V(s, r, w) = \mathrm{i}\omega\, U(s, r, \omega)$ and using (6.7) obtain

$$V(s, r, \omega) = -\omega^2 \sum_{l=1,2,3} \int_D \mathrm{d}x\, \gamma(x) f_l(x)\, w_l(\cos 2\theta_s(x, s, r))$$

$$\cdot A(x, s, r) \frac{\mathrm{N}_x \cdot \nabla\phi(x, s, r)}{4\cos^2\theta_s(x, s, r)} \mathrm{e}^{\mathrm{i}\omega\phi(x,s,r)}. \tag{6.8}$$

Using (3.7) or (3.10) as the definition of the inversion operator and setting $\tilde{c}(y) = \hat{c}(y) = c(y)$ we proceed exactly as in Section 3 to obtain instead of (3.16) the equation

$$(\mathbf{F}_m^0 f)(y) = \frac{1}{\pi^3} \int_D \mathrm{d}x \int_0^\infty \omega^2\, \mathrm{d}\omega \int_{\tilde{S}} \mathrm{d}\hat{\alpha}\, B(y, \tilde{\alpha}, \hat{\alpha}) \frac{\mathrm{N}_y \cdot \nabla\phi(y, \tilde{\alpha}, \hat{\alpha})}{4\cos^2\theta_s(y, \tilde{\alpha}, \hat{\alpha})}$$

$$\cdot \sum_{l=1,2,3} \gamma(x) f_l(x)\, g_{lm} \exp\left[\mathrm{i}\omega \frac{1}{c(y)} (\tilde{\alpha}_j + \hat{\alpha}_j)(x_j - y_j)\right], \tag{6.9}$$

where $m = 1, 2, 3$ and

$$g_{lm}(\tilde{\alpha} \cdot \hat{\alpha}, 2\theta_s) = w_l(\cos 2\theta_s(y, \tilde{\alpha}, \hat{\alpha}))\, w_m(\tilde{\alpha} \cdot \hat{\alpha}), \quad l, m = 1, 2, 3. \tag{6.10}$$

Following the derivation in Section 3 we introduce the new system of coordinates described there. We choose weight $B(y, \tilde{\alpha}, \hat{\alpha})$ as follows

$$B(y, \tilde{\alpha}, \hat{\alpha}) = \frac{1}{\text{mes } E_\psi} \frac{\cos^3 \frac{1}{2}\theta}{c^2} b(y, \theta). \tag{6.11}$$

We note that it is exactly the same weight that is described in (3.37) with $d = 1$ and $\Delta = 0$. Also we note that $d = 1$ in (3.37) corresponds to multiplying the scattered field by $\mathrm{i}\omega$ and that is exactly what we have done for the Kirchhoff approximation in (6.8).

Following the derivation in Section 3 we arrive at

$$(\mathbf{F}^0_m f)(y) = \frac{1}{(2\pi)^3} \int_D \mathrm{d}x \int_0^\infty \omega^2 \,\mathrm{d}\omega \int_{E_\theta} \mathrm{d}\theta \int_{S_\nu} \sin \tfrac{1}{2}\theta \frac{\mathrm{N}_y \cdot \nabla \phi(y, \tilde{\alpha}, \hat{\alpha})}{4 \cos^2 \theta_s}$$
$$\cdot \frac{16 \cos^4 \frac{1}{2}\theta}{c^2(y)} b(y, \theta) \sum_{l=1,2,3} \gamma(x) f_l(x)\, g_{lm} \exp\left[\mathrm{i}\omega \frac{1}{c(y)} |\tilde{\alpha} + \hat{\alpha}| \nu_j (x_j - y_j)\right], \tag{6.12}$$

where $m = 1, 2, 3$.

At this point we make use of the fact that $\mathrm{WF}(\gamma) = (x, \xi_x)$, where $\xi_x \neq 0$ is any vector in the direction of N_x, which exists because we assume that the surface is smooth so that it has a normal at every point (see Appendix D for definitions and explanations).

For every point of $\mathrm{WF}(\gamma)$ we construct a conic neighborhood (see Appendix D) and, to isolate this conic neighborhood, we define a neutralizer (or cut-off function) which is C^∞, equal to one in the conic neighborhood, and equal to zero outside some strip surrounding the conic neighborhood. We split the operator (6.12) into the sum of two operators, one with the amplitude unchanged, and another with the amplitude equal to zero, in the conic neighborhood of $\mathrm{WF}(\gamma)$. The latter operator is regularizing in the conic neighborhood by definition (see Appendix D) and thus by the theorem of Appendix D yields an infinitely differentiable function. Thus, we can restrict the amplitude in the pseudodifferential operator (6.12) to an arbitrarily small conic neighborhood of $\mathrm{WF}(\gamma)$. This in turn means that at each point y the normal N_y can be replaced by the vector ν in (3.18) and the angle θ_s by $\frac{1}{2}\theta$.

We then obtain

$$\frac{\mathrm{N}_y \cdot \nabla \phi(y, \tilde{\alpha}, \hat{\alpha})}{4 \cos^2 \theta_s} = \frac{1}{2\, c(y) \cos \frac{1}{2}\theta} \tag{6.13}$$

and the g_{lm} in (6.10) reduce to those in (3.17). We now have from (6.12)

$$(\mathbf{F}^0_m f)(y) = \frac{1}{(2\pi)^3} \int_D \mathrm{d}x \int_0^\infty \omega^2 \,\mathrm{d}\omega \int_{E_\theta} \mathrm{d}\theta \int_{S_\nu} \mathrm{d}\nu \sin \tfrac{1}{2}\theta$$
$$\cdot \frac{8 \cos^3 \frac{1}{2}\theta}{c^3(y)} b(y, \theta) \sum_{l=1,2,3} \gamma(x) f_l(x)\, g_{lm} \exp\left[\mathrm{i}\omega \frac{1}{c(y)} |\tilde{\alpha} + \hat{\alpha}| \nu_j (x_j - y_i)\right], \tag{6.14}$$

where $m = 1, 2, 3$. The expression in (6.14) is now identical to (3.23), where $\Delta = 0$ and $\tilde{c}(y) = \hat{c}(y) = c(y)$. From this point on the derivation is exactly the same as in Section 3. The only difference is that instead of (3.28) we reconstruct

$$\langle \gamma(y) f_l(y) \rangle = \frac{1}{(2\pi)^3} \int_{\Omega_\theta} \mathrm{d}p \int_D \mathrm{d}x\, \gamma(x) f_l(x)\, \mathrm{e}^{\mathrm{i}p_j(x_j - y_j)}, \quad l = 1, 2, 3, \tag{6.15}$$

but this is exactly what we want when the discontinuities of the parameters are located on smooth surfaces.

Formula (6.15) describes a bandlimited and aperture-limited reconstruction. For a smooth surface it is easy to see that (6.15) can be considered a good approximation if $\mathrm{WF}(\gamma) \cap D \times \Omega_\theta$ is not empty, which simply means that we can obtain a good bandlimited reconstruction only if we observe specular reflections from the surface.

Comparing (6.15) with (3.38) identifies the spatial operator that represents the difference between the images obtained under the assumptions of the Kirchhoff and of the Born approximations. It is clear that the separation of the parameters is not affected by this spatial operator.

Finally, we note that if we no longer assume the jumps in parameters across the interface are small, we can still apply the derivation described in this section. We set the weight $B(y, \tilde{\alpha}, \hat{\alpha})$ to be

$$B(y, \tilde{\alpha}, \hat{\alpha}) = \frac{1}{4 \operatorname{mes} E_\psi} \frac{\cos \frac{1}{2}\theta}{c^2(y)} b(y, \theta). \tag{6.16}$$

Then defining

$$\langle \gamma(y)\, R(y, \tfrac{1}{2}\theta) \rangle = \frac{1}{(2\pi)^3} \int_{\Omega_\theta} \mathrm{d}p \int_D \mathrm{d}x\, \gamma(x)\, R(x, \tfrac{1}{2}\theta)\, \mathrm{e}^{\mathrm{i}p(x-y)}, \tag{6.17}$$

we reconstruct

$$R_m(y) = \int_{E_\theta} \mathrm{d}\theta \sin \tfrac{1}{2}\theta\, w_m(\cos\theta)\, b(y, \theta)\, \langle \gamma(y)\, R(y, \tfrac{1}{2}\theta) \rangle, \quad m = 1, 2, 3. \tag{6.18}$$

Here R_m, $m = 1, 2, 3$, are weighted averages of the reflection coefficients over the range of θ, and in principle it is possible to extract the parameters.

7. Ill-conditioning of multiparameter reconstruction

Solving the linear system (3.35) to obtain the parameters requires some analysis of the numerical properties of the matrix a_{lm}. This matrix is ill-conditioned if the range of θ is small. In certain cases it may be impossible to reconstruct some of the parameters; also certain combinations of parameters are 'better' than others, i.e. some can be reconstructed with greater accuracy than others. It is of intrinsic interest to identify such combinations of parameters, given the configuration of sources and receivers.

Here, we would like to illustrate this by considering two simple examples of acoustic and P-to-P elastic scattering. Let us fix a point y at which the angle θ varies in the interval $[0, \theta_{\max}]$. In order to compute matrix a_{lm} explicitly we choose

$$b(y, \theta) = 2 \cos \tfrac{1}{2}\theta, \tag{7.1}$$

in (4.8) and (5.10), and obtain

$$a_{lm}(y) = \int_{\cos\theta_{\max}}^{1} \cos^{l+m-2}\theta \,\mathrm{d}\cos\theta = \frac{1 - \zeta^{l+m-1}}{l+m-1}, \tag{7.2}$$

where $\zeta = \cos\theta_{\max}$.

If $\theta_{\max} = \pi/2$, then $\zeta = 0$ and the matrix a_{lm} in (7.2) is a Hilbert matrix. (We note that a realistic value in a surface seismics configuration, for example, would be around $\theta_{\max} = \pi/3$.) If, on the other hand, $\theta_{\max} = 0$ then $\zeta = 1$ and the matrix a_{lm} is degenerate so that only one parameter can be recovered.

The condition number for Hilbert matrices grows exponentially with the size of the matrix. Fortunately, the matrix a_{lm} is only 2×2 for acoustic, and 3×3 for elastic scattering. As the numerical example of Section 4 demonstrates, for a reasonable range of $[0, \theta_{\max}]$, both parameters can be reconstructed.

We can find the solution explicitly: let $\varepsilon = 1-\zeta$ then from (7.2) we have

$$a_{11}=\varepsilon, \qquad a_{12}=a_{21}=\varepsilon(2-\varepsilon)/2, \qquad a_{22}=\varepsilon(\varepsilon^2-3\varepsilon+3)/3. \tag{7.3}$$

Solving (3.35) with this matrix we obtain

$$\langle f_1(y)\rangle = \frac{4(\varepsilon^2-3\varepsilon+3)}{\varepsilon^3} f_1^{\text{est}}(y) + \frac{6(\varepsilon-2)}{\varepsilon^3} f_2^{\text{est}}(y), \qquad \langle f_2(y)\rangle = \frac{6(\varepsilon-2)}{\varepsilon^3} f_1^{\text{est}}(y) + \frac{12}{\varepsilon^3} f_2^{\text{est}}(y). \tag{7.4}$$

It is easy to observe that

$$\langle f_1(y)\rangle + \langle f_2(y)\rangle = \frac{4\varepsilon-6}{\varepsilon^2} f_1^{\text{est}}(y) + \frac{6}{\varepsilon^2} f_2^{\text{est}}(y), \tag{7.5}$$

$$\langle f_1(y)\rangle - \langle f_2(y)\rangle = \frac{4\varepsilon^2-18\varepsilon+24}{\varepsilon^3} f_1^{\text{est}}(y) + \frac{6(\varepsilon-4)}{\varepsilon^3} f_2^{\text{est}}(y). \tag{7.6}$$

For small ε the right-hand side of (7.5) can be shown to be $O(1)$ as $\varepsilon\to 0$, so that $\langle f_1(y)\rangle + \langle f_2(y)\rangle$ can be recovered but $\langle f_1(y)\rangle - \langle f_2(y)\rangle$ in (7.6) cannot. It is reasonable to expect degradation of the image of the second combination as ε approaches zero.

Let us clarify the physical meaning of these combinations. Since the relative perturbations with respect to the background medium are assumed to be small, using (4.1) we can write

$$-\mathrm{d}\log(\kappa\sigma)^{-1} \approx \frac{\kappa'}{\kappa^0} + \frac{\sigma'}{\sigma^0}, \tag{7.7}$$

for $\langle f_1(y)\rangle + \langle f_2(y)\rangle$, and

$$-\mathrm{d}\log\left(\frac{\sigma}{\kappa}\right) \approx \frac{\kappa'}{\kappa^0} - \frac{\sigma'}{\sigma^0}, \tag{7.8}$$

for $\langle f_2(y)\rangle - \langle f_2(y)\rangle$. We have $-\mathrm{d}\log(\kappa\sigma)^{-1} = -2\mathrm{d}\log(c\rho)$ (twice the logarithmic derivative of the acoustic impedance) and $-\mathrm{d}\log(\sigma/\kappa) = -2\mathrm{d}\log(c)$ (twice the logarithmic derivative of the velocity). Thus, the best and the worst parameters for acoustics as $\theta_{\max}\to 0$ are logarithmic derivatives of the acoustic impedance and velocity, respectively.

To illustrate this, we consider Figs. 7 and 8 which are point-by-point sum and difference of the images in Figs. 5 and 6. According to (7.7) and (7.8) they represent the logarithmic derivatives of the acoustic impedance and the acoustic velocity, respectively. At location A in Figs. 7 and 8 the reconstruction breaks down due to lack of coverage (i.e. lack of source-receiver pairs with a specular reflection at points to the left of A). As we would expect from (7.5) and (7.6) the reconstruction for the logarithmic derivative of the acoustic velocity is more corrupted than that for the acoustic impedance.

Inversion of the single mode of P-to-P scattering involves a 3×3 matrix a_{lm}. A similar analysis identifies the following three combinations as the best, when $\theta_{\max}\to 0$,

$$\langle f_1(y)\rangle + \langle f_2(y)\rangle + \langle f_3(y)\rangle,$$

the intermediate,

$$\langle f_1(y)\rangle - \langle f_3(y)\rangle,$$

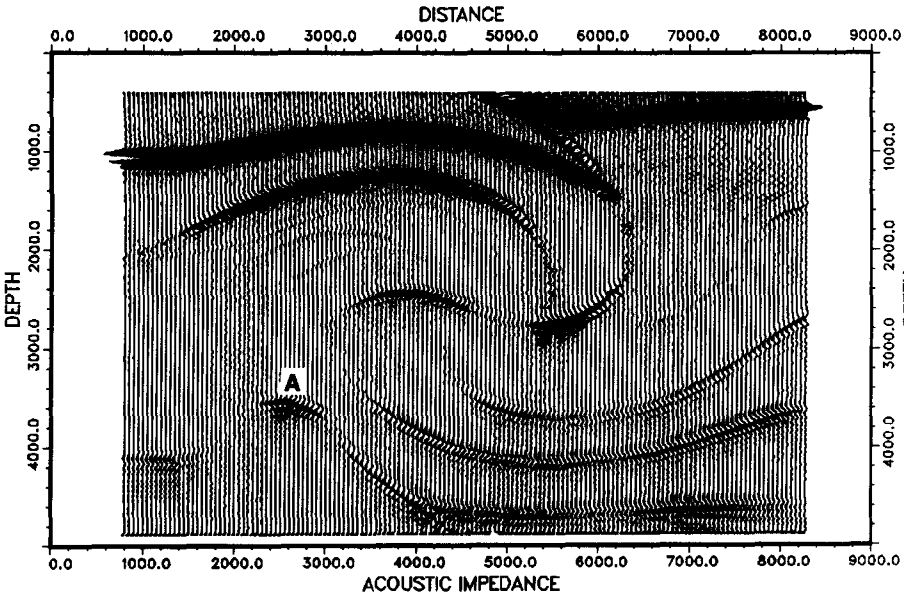

Fig. 7. Reconstruction of the acoustic impedance.

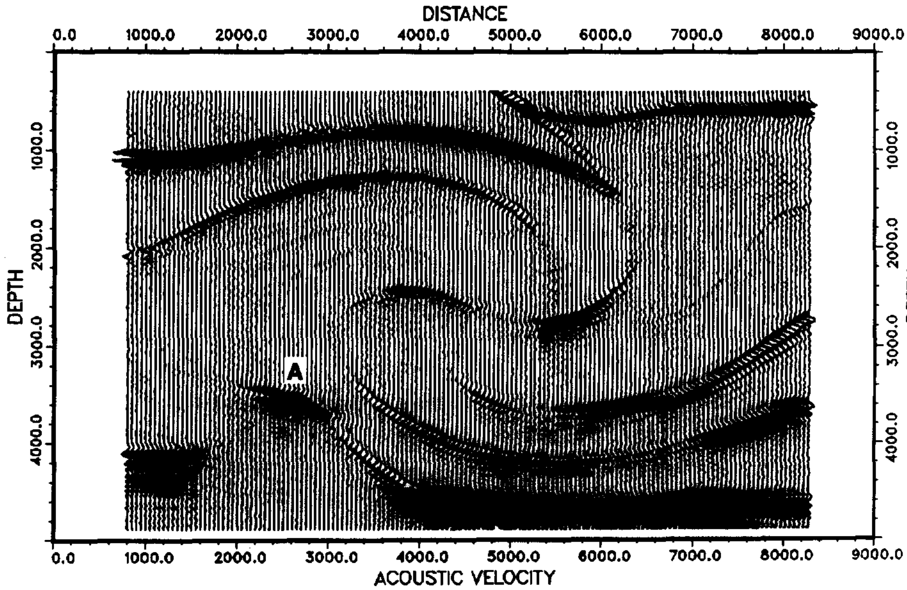

Fig. 8. Reconstruction of the acoustic velocity.

and the worst

$$\langle f_1(y)\rangle - 2\langle f_2(y)\rangle + \langle f_3(y)\rangle.$$

To clarify the physical meaning of these combinations, we write using (5.1)

$$\mathrm{d}\log(\lambda+2\mu)\rho \approx \frac{\lambda'}{\lambda^0+2\mu^0}+\frac{\rho'}{\rho^0}+\frac{2\mu'}{\lambda^0+2\mu^0}, \tag{7.9}$$

for the first combination,

$$\mathrm{d}\log(\lambda+2\mu)-4\frac{c_S^2}{c_P^2}\mathrm{d}\log\mu \approx \frac{\lambda'}{\lambda^0+2\mu^0}-\frac{2\mu'}{\lambda^0+2\mu^0}, \tag{7.10}$$

for the second, and

$$\mathrm{d}\log(\lambda+2\mu)-2\mathrm{d}\log\rho \approx \frac{\lambda'}{\lambda^0+2\mu^0}-2\frac{\rho'}{\rho^0}+\frac{2\mu'}{\lambda^0+2\mu^0} \tag{7.11}$$

for the third.

The first parameter is the logarithmic derivative of the P-wave impedance as one would expect. The other two parameters do not have specific names and are identified by formulas (7.10) and (7.11).

In general, the properties of the matrix a_{lm} can be analyzed numerically. We also note that inversion can be performed for several modes simultaneously. In this case the condition number can be expected to improve. In particular it is clear from (2.75) and (2.76) that the logarithmic derivative of the S-wave impedance $\rho'/\rho^0+\mu'/\mu^0$ can be well recovered from S-to-S scattering alone. If P-to-P and S-to-S scattering are combined into one large generalized inversion scheme at least the perturbations in the two characteristic impedances can be well reconstructed for small $\theta_{\max}$. For larger $\theta_{\max}$ P-to-S and S-to-P may contribute to enhance a third linear combination of the unknown parameters.

Concluding remarks

To summarize, we have demonstrated for acoustics and for elasticity how to solve the linearized inverse scattering problem asymptotically. The same can be done for Maxwell's equations. In fact any first order symmetric hyperbolic system arising in mathematical physics, of which these are particular examples, can be treated similarly (see [39]).

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Appendix A

In this Appendix we show that the singly scattered field can be written in the form (1.10) for acoustic media and (2.12) for elastic media when ∂D is a physical boundary. What changes is the definition of the Green's functions. Since our inversion accounts only for the most singular term, this difference in the choice of the Green's functions results in a different initial condition in the transport equations for the amplitudes.

We perform our computations in the frequency domain.

Acoustic singly scattered field in the presence of a boundary

Let the boundary ∂D be a physical boundary on which one may specify boundary conditions.

For fixed $s \in \partial D$ let $\tilde{G} = \tilde{G}(s, x, \omega)$ be the incident field which as a function of x solves the following boundary value problem

$$\omega^2 \kappa^0 \tilde{G} + (\sigma^0 \tilde{G}_{,j})_{,j} = 0, \qquad \tilde{G}_{,j} n_j|_{\partial D} = -\delta^s_{\partial D}, \quad s \in \partial D, \tag{A.1-2}$$

where (n_1, n_2, n_3) is the outer normal to ∂D. The source, which is now incorporated in the boundary condition (A.2), has the interpretation of an imposed normal particle acceleration equal to $\delta^s_{\partial D}$. The δ-function $\delta^s_{\partial D}$ on the surface is defined as follows. Let $g \in C_0^\infty(\partial D)$ be a test function. Then the distribution $\delta^s_{\partial D}$, $s \in \partial D$, is defined on $g \in C_0^\infty(\partial D)$ by the following equation:

$$\int_{\partial D} \delta^s_{\partial D} g(x) \, d\Sigma = g(s). \tag{A.3}$$

Within the single scattering approximation the scattered field $U = U(x, s, \omega)$ is the solution of the boundary value problem

$$\omega^2 \kappa^0 U + (\sigma^0 U_{,j})_{,j} = -\omega^2 \kappa' \tilde{G} - (\sigma' \tilde{G}_{,j})_{,j}, \qquad U_{,j} n_j|_{\partial D} = 0, \tag{A.4-5}$$

Let $\hat{G} = \hat{G}(x, r, \omega)$ be the solution of the boundary value problem

$$\omega^2 \kappa^0 \hat{G} + (\sigma^0 \hat{G}_{,j})_{,j} = 0, \qquad \hat{G}_{,j} n_j|_{\partial D} = -\delta^r_{\partial D}, \quad r \in \partial D, \tag{A.6-7}$$

where r denotes the receiver position on the boundary.

We multiply eq. (A.4) by $\hat{G}$ and integrate over the domain D to obtain

$$\int_D [\omega^2 \kappa^0 U \hat{G} + (\sigma^0 U_{,j})_{,j} \hat{G}] \, dx = -\int_D dx \, [\omega^2 \kappa' \tilde{G} + (\sigma' \tilde{G}_{,j})_{,j}] \hat{G}. \tag{A.8}$$

Integrating the second term on the left by parts twice and using eq. (A.6) for $\hat{G}$ we obtain

$$\int_{\partial D} \sigma^0 [U_{,j} n_j \hat{G} - \hat{G}_{,j} n_j U] \, d\Sigma = -\int_D dx \, [\omega^2 \kappa' \tilde{G} + (\sigma' \tilde{G}_{,j})_{,j}] \hat{G}. \tag{A.9}$$

Then, using boundary conditions (A.5) for U and (A.7) for $\hat{G}$, we arrive at

$$U(s, r, \omega) = -\int_D dx \, [\omega^2 \kappa' \tilde{G} + (\sigma' \tilde{G}_{,j})_{,j}] \hat{G}. \tag{A.10}$$

Integrating the second term on the right by parts and noticing that the boundary term vanishes since the perturbation σ' is zero on ∂D, we obtain

$$U(s, r, \omega) = -\int_D dx \, [\omega^2 \kappa' \tilde{G} \hat{G} - \sigma' \tilde{G}_{,j} \hat{G}_{,j}], \tag{A.11}$$

which is the integral representation of the singly scattered field. The Fourier transform of this representation should be compared with (1.10).

The elastic singly scattered field in the presence of a boundary

Let $\tilde{G}_{jl} = \tilde{G}_{jl}(s, x, \omega)$ be the incident field, which solves the following boundary value problem

$$\rho^0 \omega^2 \tilde{G}_{jl} + (c^0_{lmpq} \tilde{G}_{jp,q})_{,m} = 0, \qquad c^0_{lmpq} \tilde{G}_{jp,q} n_m = -\delta_{jl} \delta^s_{\partial D}. \tag{A.12-13}$$

Here $\tilde{G}_{jl}$ is the displacement in the l-direction at x due to a point force in the j-direction at s on the boundary. Within the single scattering approximation the scattered field $U_{jl} = U_{jl}(s, x, \omega)$ is the solution of the boundary value problem

$$\rho^0 \omega^2 U_{jl} + (c^0_{lmpq} U_{jp,q})_{,m} = -\rho' \omega^2 \tilde{G}_{jl} - (c'_{lmpq} \tilde{G}_{jp,q})_{,m}, \qquad c^0_{lmpq} U_{jp,q} n_m = 0. \tag{A.14-15}$$

Let $\hat{G}_{kl} = \hat{G}_{kl}(x, r, \omega)$ satisfy the following boundary value problem

$$\rho^0 \omega^2 \hat{G}_{kl} + (c^0_{lmpq} \hat{G}_{kp,q})_{,m} = 0, \qquad c^0_{lmpq} \hat{G}_{kp,q} n_m = -\delta_{kl} \delta^r_{\partial D}, \tag{A.16-17}$$

where r denotes the receiver position.

We multiply eq. (A.14) by the solution $\hat{G}_{kl}$ of the boundary value problem (A.16) and (A.17), sum over index l, and integrate over the domain D to obtain

$$\int_D [\rho^0 \omega^2 U_{jl} \hat{G}_{kl} + (c^0_{lmpq} U_{jp,q})_{,m} \hat{G}_{kl}] \, dx = -\int_D [\rho' \omega^2 \tilde{G}_{jl} \hat{G}_{kl} + (c'_{lmpq} \tilde{G}_{jp,q})_{,m} \hat{G}_{kl}] \, dx. \tag{A.18}$$

Integrating the second term on the left by parts twice and changing the dummy index l to p in the first term we have

$$\int_D [\rho^0 \omega^2 \hat{G}_{kp} U_{jp} + (c^0_{lmpq} \hat{G}_{kl,m})_{,q} U_{jp}] \, dx + \int_{\partial D} [c^0_{lmpq} U_{jp,q} n_m \hat{G}_{kl} - c^0_{lmpq} \hat{G}_{kl,m} n_q U_{jp}] \, d\Sigma$$
$$= -\int_D [\rho' \omega^2 \tilde{G}_{jl} \hat{G}_{kl} + (c'_{lmpq} \tilde{G}_{jp,q})_{,m} \hat{G}_{kl}] \, dx. \tag{A.19}$$

Using the symmetry relation $c^0_{lmpq} = c^0_{pqlm}$ we notice that the volume integral on the left-hand side of eq. (A.19) vanishes due to (A.16), and we obtain

$$\int_{\partial D} [c^0_{lmpq} U_{jp,q} n_m \hat{G}_{kl} - c^0_{pqlm} \hat{G}_{kl,m} n_q U_{jp}] \, d\Sigma = -\int_D [\rho' \omega^2 \tilde{G}_{jl} \hat{G}_{kl} + (c'_{lmpq} \tilde{G}_{jp,q})_{,m} \hat{G}_{kl}] \, dx. \tag{A.20}$$

Using boundary conditions (A.15) for U_{jl} and (A.17) for $\hat{G}_{kl}$ we arrive at

$$U_{jk}(s, r, \omega) = -\int_D [\rho' \omega^2 \tilde{G}_{jl} \hat{G}_{kl} + (c'_{lmpq} \tilde{G}_{jp,q})_{,m} \hat{G}_{kl}] \, dx. \tag{A.21}$$

Integrating the second term on the right by parts, and noticing that the boundary term vanishes because the perturbations c'_{lmpq} and zero on ∂D, we obtain

$$U_{jk}(s, r, \omega) = -\int_D [\rho' \omega^2 \tilde{G}_{jl} \hat{G}_{kl} - c'_{lmpq} \tilde{G}_{jp,q} \hat{G}_{kl,m}] \, dx, \tag{A.22}$$

which is an integral representation of the singly scattered field. Here U_{jk} is the k-component of the scattered field at r due to the point force in the j-direction at s. The Fourier transform of this representation should be compared with (2.12).

Appendix B

Part 1

The argument that it is sufficient to consider the leading order term in the forward problem to account for the most singular term in the linearized inverse problem is presented in [40] for the reduced wave equation. We describe here the main points of this argument.

In the acoustic case, instead of simplifying (1.7) to obtain (1.21), we may choose to rewrite (1.7) in the frequency domain as

$$U(s,r,\omega)=-\int_D \mathrm{d}x\,[\omega^2\kappa'\tilde{G}\hat{G}-\sigma'\tilde{G}_{,j}\hat{G}_{,j}]. \tag{B.1}$$

We then construct the operator

$$(\mathbf{R}_m^* U)(y)=-\frac{1}{\pi^2}\int_{\partial D}\mathrm{d}s\int_{\partial D}\mathrm{d}r\int_{-\infty}^{+\infty}\mathrm{d}\omega\,\frac{\bar{\tilde{G}}(s,y,\omega)\,\bar{\hat{G}}(y,r,\omega)}{|\tilde{G}|^2\,|\hat{G}|^2}\,h_m(s,r,y)\,U(s,r,\omega), \tag{B.2}$$

with $m=1,2$.

In this formula $\bar{\tilde{G}}$ and $\bar{\hat{G}}$ are complex conjugates of $\tilde{G}$ and $\hat{G}$ and $h_m(s,r,y)$ is yet to be defined. If we substitute (B.1) into (B.3) we will obtain a Fourier integral operator, $\mathbf{F}_m$ say, applied to the perturbations in the acoustic parameters $\boldsymbol{f}=(\kappa'/\kappa^0,\sigma'/\sigma^0)$:

$$(\mathbf{F}_m\boldsymbol{f})(y)=\frac{1}{\pi^2}\int_{\partial D}\mathrm{d}s\int_{\partial D}\mathrm{d}r\int_{-\infty}^{+\infty}\mathrm{d}\omega\int_D\mathrm{d}x$$
$$\cdot\frac{\bar{\tilde{G}}(y)\bar{\hat{G}}(y)}{|\tilde{G}(y)|^2|\hat{G}(y)|^2}\,h_m(s,r,y)\,(\omega^2\kappa'\tilde{G}(x)\hat{G}(x)-\sigma'\tilde{G}_{,j}(x)\hat{G}_{,j}(x)), \tag{B.3}$$

where we have written only those arguments necessary to avoid confusion.

We now notice that if we restrict the integration with respect to frequency ω in (B.3) to any bounded interval the result will be infinitely differentiable with respect to y. This follows from the fact that we can perform the differentiation under the integral sign as long as the total domain of integration is compact. Since we are interested only in the leading term with respect to smoothness in the inversion, it is sufficient to consider only large ω. Thus, we can use high-frequency asymptotic approximations to the Green's functions in (B.3) and, hence, in *both* (B.1) and (B.2). Therefore, further analysis can be restricted to the leading order terms, and that is what we do in the main body of the paper. The same considerations apply to the elastic case.

The choice of the weight function $h_m(s,r,y)$ is described in the main body of the paper.

Part 2

For definiteness let us consider the Jacobian $J(y,r)$. This Jacobian arises from the change of variables

$$\alpha_l=c(y)\,\phi_{,l}(y,r),\quad l=1,2,3, \tag{B.4}$$

where $\phi(y,r)$ is the travel time between points y and r and $c(y)$ is the velocity at y.

As defined in (3.9) this Jacobian is the ratio of two elementary areas. Let us denote by $\theta_r(y,r)$ the angle that the ray connecting r and y makes with the vertical at r on the surface. Let $\mathrm{d}S(y,r)$ be the elementary cross-sectional area at r of the ray tube associated with such a ray. Then

$$\cos\theta_r\,\mathrm{d}r=\mathrm{d}S, \tag{B.5}$$

and, therefore,

$$J(y, r) = \cos \theta_r \frac{\mathrm{d}\alpha}{\mathrm{d}S}. \tag{B.6}$$

We note that $\mathrm{d}\alpha/\mathrm{d}S$ is the reciprocal of the geometrical spreading associated with such a ray tube, which also appears in the computation of the amplitudes by the ray method. Thus, no extra computational effort is required to compute J.

Part 3

Given (3.13), our goal is to derive the most singular terms in the asymptotics of the operators $\mathbf{F}_m$, $m = 1, 2, 3$. Consider the function

$$\Phi(x, y, s, r) = \phi(x, s, r) - \phi(y, s, r). \tag{B.7}$$

The behavior of the Fourier integral operators in (3.13) is determined by the set

$$C_\Phi = \{(\omega, s, r, x, y) \in R_+ \times \partial D \times \partial D \times D \times D\colon \Phi(x, y, s, r) = 0, \\ \nabla_s \Phi(x, y, s, r) = 0, \nabla_r \Phi(x, y, s, r) = 0\}. \tag{B.8}$$

This definition transforms into the standard one under the change of variables $(\omega, s, r) \to (p, \theta, \psi)$ which is described in detail in Section 3. Here $p = \omega'\nu \in R^3$, see (3.28), and θ, ψ are angles. For every fixed θ and ψ the standard definition of the set C_Φ is as follows

$$C_\Phi = \{(p, x, y) \in R^3 \times D \times D\colon \nabla_p[\omega(p, \theta, \psi)\, \Phi(x, y, s(p, \theta, \psi), r(p, \theta, \psi))] = 0\}, \tag{B.9}$$

and (B.9) can be shown to be equivalent to (B.8).

On the other hand one can verify that under the assumption formulated in [4], namely that for each point in D the map between directions at the point and the boundary ∂D is a diffeomorphism, the set C_Φ is

$$C_\Phi = \{(\omega, s, r, x, x) \in R_+ \times \partial D \times \partial D \times D \times D\}, \tag{B.10}$$

so that its projection on $D \times D$ is the diagonal. This implies (see e.g. [42]) that the operators $\mathbf{F}_m$, $m = 1, 2, 3$, are pseudodifferential operators and we can restrict the integration in (3.13) to the neighborhood of the diagonal since this will modify the operators $\mathbf{F}_m$, $m = 1, 2, 3$, only by some regularizing operator.

Appendix C

This result and some of its implications are discussed in Burridge and Beylkin (1987). We describe it here for completeness. In the body of the paper we use it for the spatial dimension $n = 3$, $\xi = \tilde{\alpha}$ and $\eta = \hat{\alpha}$.

Lemma. *Consider the differential forms* $\mathrm{d}\xi$, $\mathrm{d}\eta$, $\mathrm{d}\nu$ *on unit* $(n-1)$*-dimensional spheres over which the corresponding vectors vary. Then*

$$\mathrm{d}\xi\, \mathrm{d}\eta = \sin^{n-2}\theta\, \mathrm{d}\theta\, \mathrm{d}\psi\, \mathrm{d}\nu, \tag{C.1}$$

where ψ *varies over an* $(n-2)$*-dimensional sphere and* $\theta \in [0, \pi]$. *The unit vector* ψ *lies in the plane of* ξ, η, ν *and is perpendicular to* ν:

$$\xi = \cos\alpha\, \nu - \sin\alpha\, \psi, \qquad \eta = \cos\beta\, \nu + \sin\beta\, \psi, \qquad \theta = \alpha + \beta, \tag{C.2}$$

with α *and* β *functions of* θ.

Proof. Let ψ be a unit vector perpendicular to ν and lying in the plane of ξ, η. Choose coordinates so that ν lies along the x_n axis, and ξ, η, ν, ψ lie in the (x_{n-1}, x_n)-plane. Then from (C.2) it is clear that

$$\left.\begin{aligned} d\xi_k &= \cos\alpha\, d\nu_k - \sin\alpha\, d\psi_k \\ d\eta_k &= \cos\beta\, d\nu_k + \sin\beta\, d\psi_k \end{aligned}\right\}, \quad \text{for } k=1,\dots,n-2, \tag{C.3}$$

and so

$$d\xi_k\, d\eta_k = \sin\theta\, d\nu_k\, d\psi_k, \quad \text{for } k=1,\dots,n-2. \tag{C.4}$$

Here we have used (C.3) and the addition formula for sine.

In the (x_{n-1}, x_n)-plane let us denote by $d\xi'$, $d\nu'$, $d\eta'$, $d\psi'$ the infinitesimal angular displacements of those vectors. Then

$$d\xi' = d\nu' - d\alpha, \qquad d\eta' = d\nu' + d\beta, \tag{C.5}$$

and so

$$d\xi'\, d\eta' = d\nu'(d\alpha + d\beta) = d\nu'\, d\theta, \tag{C.6}$$

Thus, on combining (C.4) and (C.6) we have

$$d\xi\, d\eta = d\nu\, d\psi\, \sin^{n-2}\theta\, d\theta, \tag{C.7}$$

where

$$\begin{aligned} &d\xi = d\xi_1 \cdots d\xi_{n-2}\, d\xi', \quad d\eta = d\eta_1 \cdots d\eta_{n-2}\, d\eta', \\ &d\nu = d\nu_1 \cdots d\nu_{n-2}\, d\nu', \qquad d\psi = d\psi_1 \cdots d\psi_{n-2}. \end{aligned} \tag{C.8}$$

Appendix D: wave front sets and microlocalization

The study of singularities of distributions is greatly clarified if one considers simultaneously the location of the singularity and the 'direction' in which the singularity occurs. In linearized inverse problems which use the Kirchhoff approximation such an approach allows us to justify the transformations that we perform in deriving the term most singular with respect to smoothness. For completeness we present here some definitions and a theorem. For details see e.g. Hörmander [41] or Treves [42]. Notations are local to this appendix.

Let Ω be some open domain of R^n. Let us consider elements of $\Omega \times R^n$, (x_0, ξ^0) where $\xi^0 \neq 0$. By a conic neighborhood of (x_0, ξ^0) we understand a set $U_0 \times K^0$ where U_0 is an open set containing x_0 and K^0 is an open cone in R^n containing ξ^0.

Let us denote the Fourier transform by $\hat{}$ and the space of C^∞ functions having compact support in Ω by $C_0^\infty(\Omega)$.

Definition 1. A distribution u in Ω is C^∞ in the conic neighborhood of a point $(x_0, \xi^0) \in \Omega \times (R^n \backslash \{0\})$ if there is a function $g \in C_0^\infty(\Omega)$ equal to one in U_0 such that for every $M \geq 0$ there is a constant C_M such that

$$|(\hat{gu})(\xi)| \leq C_M (1 + |\xi|)^{-M}, \quad \text{for every } \xi \in K^0. \tag{D.1}$$

Definition 2. A distribution u is C^∞ in the conic open subset of $\Omega\times(R^n\backslash\{0\})$ if it is C^∞ in a conic neighborhood of every point of the subset.

Definition 3. The complement in $\Omega\times(R^n\backslash\{0\})$ of the union of all conic open sets in which the distribution is C^∞ is called the wave-front set $\mathrm{WF}(u)$ of the distribution u.

Consider a pseudodifferential operator A

$$(\mathrm{A}u)(x)=\int \mathrm{d}\xi\, a(x,\xi)\,\hat{u}(\xi)\,\mathrm{e}^{\mathrm{i}\xi\cdot x}, \tag{D.2}$$

where $a(x,\xi)$ is the standard symbol of the operator A.

Definition 4. A pseudodifferential operator A in Ω is regularizing in the conic neighborhood $U_0\times K^0$ of a point $(x_0,\xi^0)\in\Omega\times R^n$ if there is a function $g\in C_0^\infty(\Omega)$ equal to one in U_0 such that for every $M\geq 0$ and every pair of multi-indices α and β there is a constant $C_{M,\alpha,\beta}$ such that

$$\sup_x|\partial_x^\alpha\partial_\xi^\beta[g(x)a(x,\xi)]|\leq C_{M,\alpha,\beta}(1+|\xi|)^{-M},\quad \text{for every } \xi\in K^0. \tag{D.3}$$

Definition 5. A pseudodifferential operator A in Ω is regularizing in a conic open subset of $\Omega\times(R^n\backslash\{0\})$ if it is regularizing in a conic neighborhood of every point of the subset.

Definition 6. The complement in $\Omega\times(R^n\backslash\{0\})$ of the union of all conic open sets in which the pseudodifferential operator A is regularizing is called the microsupport of A and is denoted by $\mu\mathrm{supp}\,\mathrm{A}$.

Theorem. *Let* A *be a properly supported pseudodifferential operator in Ω and u a distribution in Ω. Then*

$$\mathrm{WF}(\mathrm{A}u)\subset\mathrm{WF}(u)\cap\mu\mathrm{supp}\,\mathrm{A}. \tag{D.4}$$

It follows from (D.4) that if $\mathrm{WF}(u)\cap\mu\mathrm{supp}\,\mathrm{A}$ is empty then $\mathrm{A}u\in C^\infty(\Omega)$.

In linearized inverse problems which use the Kirchhoff approximation we choose, following Bleistein [33], to reconstruct the singular function $\gamma(x)$ of the smooth surface Γ which is defined by

$$\int_{R^3}\gamma(x)f(x)\,\mathrm{d}x=\int_\Gamma f(\sigma)\,\mathrm{d}\sigma, \tag{D.5}$$

where f spans an appropriate class of test functions.

Let the unit vector N_x be normal to the surface at the point x. Then

$$\mathrm{WF}(\gamma)=\{(x,\lambda\mathrm{N}_x)\colon x\in\Gamma \text{ and } \lambda\neq 0\in R\}. \tag{D.6}$$

Glossary

Symbol	*First occurrence*	*Definition*
a_1, a_2	(2.52)	scalar coefficients in S polarization
$a_{lm}(y)$	(3.35)	matrix in generalized linear inversion
$\tilde{A}(s, x)$	(1.11)	amplitude of $\tilde{G}$
$\hat{A}(x, r)$	(1.11)	amplitude of $\hat{G}$
$B(y, s, r) = B(y, \tilde{\alpha}, \hat{\alpha})$	(3.7)	adjustable weighting function
c	(1.1)	wave speed
$c^0 = \sqrt{\sigma^0/\kappa^0}$	(1.1)	background wave speed
$c_{lmpq} = c_{lmpq}(x)$	(2.1)	elastic stiffness tensor at the point x
D	(1.4)	domain in x-space
∂D	(1.4)	boundary of D
E_θ	(3.18)	range of θ
E_ψ	(3.21)	range of ψ
$f(x) = (f_1(x), f_2(x), f_3(x))$	(3.1)	vector function of x
$f_m^{\text{est}}(y)$	(3.33)	estimated reconstruction, pre-image
$\mathbf{F}_m, \mathbf{F}_m^0$	(3.13), (3.16)	Fourier integral operator
$g_{lm} = w_l w_m$	(3.17)	matrix in generalized linear inversion
$\hat{G} = \hat{G}(x, r, t)$	(1.7)	Green's function
$\tilde{G} = \tilde{G}(s, x, t)$	(1.4)	Green's function
j	(1.1)	integer index
$_{,j}$	(1.1)	derivative with respect to x_j
$J = J(y, s)$ or $J(y, r)$	(3.7)	Jacobian
p	(1.1), (3.30)	pressure, or (unrelated) $\omega \nabla_x \phi(x, s, r)$
^{P} or $_P$	(2.17)	pertaining to P-waves
^{PP}	(2.33)	pertaining to P-to-P scattering
^{PS}	(2.33)	pertaining to P-to-S scattering
r	(1.7)	receiver position
Re	(3.14)	real part
$\mathbf{R}$	(3.1)	generalized Radon transform
$\mathbf{R}_m^*$	(3.7)	generalized backprojection
s	(1.1)	source position
$\tilde{S}$ or $\hat{S}$	(3.10)	domains of integration on unit sphere
S^2	(3.10)	unit sphere in 3-space
^{S} or $_S$	(2.17)	pertaining to S-waves
^{SP}	(2.33)	pertaining to S-to-P scattering
^{SS}	(2.33)	pertaining to S-to-S scattering
t	(1.1)	time
$\mathbf{T}_m$	(3.32)	operator of class $L^{-1}(D)$
$u_l = u_l(x, t)$	(2.1)	l-component of the displacement vector
$U = U(x, s, t)$	(1.6)	single scattering solution
$U_{jk}(s, r, t)$	(2.8)	k-component of the scattered field at r due to a point force in the j-direction at s
V	(3.6)	second derivative of the Radon transform of f

w_l	(3.1)	weighting functions
$x=(x_1, x_2, x_3)$	(1.1)	cartesian position vector
$\alpha=(\alpha_1, \alpha_2, \alpha_3)$	(2.38)	unit tangent to ray
$\beta=(\beta_1, \beta_2, \beta_3)$	(2.50)	unit vector perpendicular to ray
$\gamma=(\gamma_1, \gamma_2, \gamma_3)$	(2.51)	unit vector perpendicular to ray
$\Delta(y)$	(3.19)	dimensionless parameter
δ	(1.4)	Dirac delta
δ_{ij}	(1.4)	Kronecker delta
$\theta=\theta(x, s, r)$	(1.19)	angle between rays at x
$\kappa(x)$	(1.1)	compressibility
λ	(2.2)	Lamé constant
μ	(2.2)	Lamé constant
$\rho=\rho(x)$	(2.1)	density
$\sigma(x)$	(1.1)	specific volume
$\nu=(\nu_1, \nu_2, \nu_3)$	(3.18)	unit vector at point of reflection
$\hat{\phi}(s, x)$	(1.12)	phase of $\hat{G}$
$\tilde{\phi}(s, x)$	(1.11)	phase of $\tilde{G}$
$\phi(x, s, r)=\tilde{\phi}(s, x)+\hat{\phi}(x, r)$	(3.5)	two-way travel time (phase)
ω	(3.11)	angular frequency
Ω_θ	(3.29)	domain of integration
∂_t	(1.1)	derivative with respect to t
$*_t$	(1.9)	convolution in t
^{0}	(1.2)	pertaining to the background
$'$	(1.2)	pertaining to the perturbation
$\hat{}$	(1.7)	pertaining to r
$\tilde{}$	(1.4)	pertaining to s
$\bar{}$	(3.12)	complex conjugate
$\langle\cdot\rangle$	(3.28)	band-restricted reconstruction
$\|\cdot\|$	(5.6)	Euclidean vector norm

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