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Iterated Spherical Means in Linearized Inverse Problems

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Abstract. We consider a representation of the function

$$F(x) = \frac{1}{(2\pi)^n} \int_{|p|<2k} \hat{F}(p) e^{ip\cdot x} dp,$$

in terms of the iterated spherical mean of $\hat{F}(p)$. Here, n is the dimension of the space. We also review applications of such a representation to linearized inverse problems and present as examples problems of diffraction tomography and inverse scattering in Born (and Rytov) approximations.

1. Introduction. Experiments in scattering usually yield the measured scattered field as a function of two unit vectors which represent the direction of propagation of the incident wave and the direction at which the field is recorded. In many cases (we provide two examples in this article) if we fix the direction of the incident wave then what we obtain in a single experiment is the Fourier transform of the quantity we would like to recover restricted to some sphere (or circle in the two-dimensional case). Recently algorithms which make use of such data were suggested by A. J. Devaney (see Refs. [1-3]). These algorithms which solve linearized inverse problems are based on the representation of a function in terms of the iterated spherical mean of its Fourier transform. A uniform derivation of such a representation independent of the dimension of the space is presented in Ref. [4].

In this article we briefly describe the derivation of the representation of a function in terms of the iterated spherical mean of its Fourier transform and consider applications of this representation to linearized inverse problems. We treat problems of inverse scattering and diffraction tomography. In the case of diffraction tomography our consideration differs from one presented in Ref. [1].

2. Spherical Means, Iterated Spherical Means and the Fundamental Identity. Let f be a continuous function in R^n. The spherical mean of the function f is defined as

$$I(x,r) = \frac{1}{\omega_n} \int_{|\nu|=1} f(x + r\nu) d\omega_\nu, \tag{2.1}$$

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where $\omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ is the surface area of the unit sphere in R^n, x is a point in R^n, ν is a unit vector in R^n and $d\omega_\nu$ is the standard measure on the unit sphere (the solid angle differential form), such that $\int_{|\nu|=1} d\omega_\nu = \omega_n$. The function I(x,r) is the normalized average of the function f on a sphere of radius |r| about the point x. We note that the function I(x,r) is even with respect to r.

The iterated spherical mean $M(x,\alpha,\beta)$ is defined as follows

$$M(x,\alpha,\beta) = \frac{1}{\omega_n^2} \int_{|\mu|=1} \int_{|\nu|=1} f(x + \alpha\mu + \beta\nu) d\omega_\nu d\omega_\mu, \tag{2.2}$$

where α,β are real numbers and ν,μ are unit vectors in R^n. F. John [5] obtained the fundamental identity

$$M(x,\alpha,\beta)= \frac{2\omega_{n-1}}{(2\alpha\beta)^{n-2}\omega_n} \int_{\beta-\alpha}^{\beta+\alpha} [(r+\beta-\alpha)(r+\beta+\alpha)(\alpha+r-\beta)(\alpha-r+\beta)]^{(n-3)/2} r I(x,r)dr, \tag{2.3}$$

which relates the iterated spherical mean of a function to the spherical mean of that function.

3. The Representation of a Function in Terms of the Iterated Spherical Mean of its Fourier Transform. The fundamental identity in (2.3) can be used (see [4]) to obtain the representation of a function in terms of the iterated spherical mean of its Fourier transform. Let us briefly describe the derivation.

We consider the function

$$f_y(p) = \frac{|p|}{(4k^2-|p|^2)^{(n-3)/2}} \hat{F}(p) e^{ip\cdot y}, \tag{3.1}$$

where p and y belong to R^n and the function $\hat{F}(p)$ has support inside the n-dimensional ball

$$B_{2k} = \{p: |p| < 2k\}. \tag{3.2}$$

Let vector y in (3.1) be a parameter. First, we compute the spherical mean $I_y(0,r)$ of the function $f_y(p)$. Then we compute the iterated spherical mean of the function $f_y(p)$, the function $M(0,k,k)$, using the fundamental identity in (2.3) and the definition in (2.2). Comparing the results we obtain the following representation

$$F(x) = \frac{k^n}{8\pi^n \omega_{n-1}} \int_{|\mu|=1} \int_{|\nu|=1} \frac{|\nu - \mu|}{(4-|\nu-\mu|^2)^{(n-3)/2}} \hat{F}(k\nu-k\mu) e^{i(k\nu-k\mu)\cdot x} d\omega_\nu d\omega_\mu, \tag{3.3}$$

where

$$F(x) = \frac{1}{(2\pi)^n} \int_{|p|<2k} \hat{F}(p) e^{ip\cdot x} dp. \tag{3.4}$$

If (as we assumed initially) the support of the function $\hat{F}$ is contained in the ball B_{2k} described in (3.2) then the function F(y) defined in (3.4) coincides with the inverse Fourier transform of $\hat{F}(p)$. The identity in (3.3) in this case is the representation of a function in terms of the iterated spherical mean of its Fourier transform.

If the support of $\hat{F}(p)$ is not restricted to the ball B_{2k} in (3.2), then (3.4) defines the low-pass-filtered version of the function whose Fourier transform is $\hat{F}(p)$. In this case (3.3) is the representation of the low-pass-filtered version of the function in terms of the iterated spherical mean of its Fourier transform.

The representation in (3.3) was derived in [4]. For $n = 2$ and $n = 3$ it reduces to formulae obtained by A. J. Devaney [1,3].

4. Inverse Scattering in Born Approximations. The most simple example of an application of the representation in (3.3) is the inversion formula for inverse scattering within Born approximation [3]. Let us consider the three-dimensional case for simplicity and let $e^{ik\nu\cdot x}$ (a plane wave) be the incident field. Consider the wave function $\Psi(x,k,\nu)$ which satisfies the Lippmann-Schwinger integral equation

$$\Psi(x,k,\nu) = e^{ik\nu\cdot x} - \frac{1}{4\pi}\int \frac{e^{ik|x-y|}}{|x-y|} V(y)\, \Psi(y,k,\nu)\, dy, \tag{4.1}$$

where the potential V is such that for large $|x|$ the solution of (4.1) has the asymptotics

$$\Psi(x,k,\nu) = e^{ik\nu\cdot x} + \frac{e^{ik|x|}}{|x|} f(k,\nu,\mu) + o\left(\frac{1}{|x|}\right),$$

where $\mu = \frac{x}{|x|}$.

If the solution $\Psi(x,k,\nu)$ of the integral equation in (4.1) is known then the scattering amplitude $f(k,\nu,\mu)$ can be written as follows

$$f(k,\nu,\mu) = -\frac{1}{4\pi}\int e^{-ik\mu\cdot x} V(x)\, \Psi(x,k,\nu)\, dx. \tag{4.2}$$

Using Born approximation by setting $\Psi(x,k,\nu) = e^{ik\nu\cdot x}$ in (4.2) we linearize the relation between the potential and the scattering amplitude and obtain

$$f(k,\nu,\mu) = -\frac{1}{4\pi}\int e^{-ik\mu\cdot x} V(x) e^{ik\nu\cdot x} dx = -\frac{1}{4\pi}\hat{V}(k\mu - k\nu). \tag{4.3}$$

Similarly, we derive that within Born approximation

$$|f(k,\nu,\mu)|^2 = \frac{1}{(4\pi)^2}\hat{Q}(k\mu - k\nu), \tag{4.4}$$

where $\hat{Q}$ is the Fourier transform of the function

$$Q(x) = \int V(x+y)\, V^*(y)\, dy.$$

(The function $Q(x)$ is the so-called interatomic distance function).

Making use of the representation in (3.3), where we set the dimension $n = 3$, we obtain

$$V_{LP}(x) = -\frac{k^3}{4\pi^3}\int_{|\mu|=1}\int_{|\nu|=1} |\nu-\mu|\, f(k,\nu,\mu) e^{i(k\nu - k\mu)\cdot x} d\omega_\nu\, d\omega_\mu, \tag{4.5}$$

and

$$Q_{LP}(x) = \frac{k^3}{\pi^2} \int_{|\mu|=1} \int_{|\nu|=1} |\nu-\mu| |f(k,\nu,\mu)|^2 e^{i(k\nu-k\mu)\cdot x} d\omega_\nu d\omega_\mu . \quad (4.6)$$

We note, that the sphere of radius 2k which contains the ball B_{2k} in (3.2) is the so-called Ewald limiting sphere.

Thus, we obtain that if we can measure the phase of the scattering amplitude we have the explicit inversion formula in (4.5) for the reconstruction of the low-pass-filtered version of the potential. In the case when the phase of the scattering amplitude cannot be directly measured we can explicitly reconstruct the interatomic distance function using (4.6). Formulae (4.5) and (4.6) were first obtained in Refr. [3].

Remark 1: In the case of inverse scattering in the n-dimensional space one obtains the analogous result as soon as the scattering amplitude is properly defined.

Remark 2: We can always write

$$V(x) = \lim_{k\to\infty} - \frac{k^3}{4\pi^3} \int_{|\mu|=1} \int_{|\eta|=1} |\nu - \mu| f(k,\nu,\mu) e^{i(k\nu-k\mu)\cdot x} d\omega_\nu d\omega_\mu$$

This statement is equivalent to the uniqueness theorem (if we know the scattering amplitude for large k).

5. Diffraction Tomography in Born (and Rytov) Approximations. We use the inhomogeneous Helmholtz equation to describe the wave propagation; namely, we consider

$$(\Delta + k^2)U(x,k) = k^2 O(x)\, U(x,k), \quad (5.1)$$

where

$$O(x) = 1 - n^2(x).$$

The Helmholtz equation in (5.1) describes the acoustic field in a fluid medium. The parameter $k = 2\pi/\lambda$ is the wavenumber (here λ is a wavelength). We call the function $O(x)$ an object profile. We assume that the index of refraction

$$n(x) = 1,$$

if $|x| > R$ for some $R > 0$. It means that the support of the object profile $O(x)$ is contained within the ball $B_R = \{x: |x| < R\}$.

The inverse problem of diffraction tomography consists of determining the object profile $O(x)$ from the scattered acoustic field measured outside the ball B_R. This problem is nonlinear. We will use the first Born approximation to obtain a linear relation between the scattered field and the object profile. We consider the two-dimensional case for simplicity.

We start with the integral equation for the Fourier transform of the function $U(x,k)$ in (5.1),

$$\Psi(p,k,\nu) = \delta(p-k\nu) + \frac{k^2}{k^2-|p|^2-i0} \int \hat{O}(p-p')\, \Psi(p',k,\nu)dp', \quad (5.2)$$

where

$$\Psi(p,k,\nu) = \int U(x,k)e^{-ip\cdot x}dx,$$

$\delta(p-k\nu)$ represents the incident plane wave, p is a vector in R^2, ν is a unit vector in R^2 and

$$\hat{O}(p) = \int O(x)e^{-ip\cdot x}dx.$$

We introduce a system of coordinates which is related to the direction of propagation of the initial plane wave. We set

$$p = \eta\nu + \xi\nu^{\perp}, \tag{5.3}$$

where $\nu^{\perp}$ is the unit vector orthogonal to the vector ν: $\nu = (\nu_1,\nu_2)$ and $\nu^{\perp} = (-\nu_2,\nu_1)$.

Let us consider the scattered field $\Psi_{sc}(p,k,\nu) = \Psi(p,k,\nu) - \delta(p-k\nu)$. We find

$$\Psi_{sc}(p,k,\nu) = \frac{k^2}{k^2 - |p|^2 - i0} \int \hat{O}(p-p')\,\Psi(p',k,\nu)dp', \tag{5.4}$$

and in the system of coordinates (5.3) we have

$$\Psi_{sc}(\eta\nu+\xi\nu^{\perp},k,\nu) = \frac{k^2}{k^2-\xi^2-\eta^2-i0} \int \hat{O}(\eta\nu+\xi\nu^{\perp}-p')\,\Psi(p',k,\nu)dp'. \tag{5.5}$$

We take the inverse Fourier transform of the function Ψ_{sc} in η-coordinate

$$\Psi_{sc}(y,\xi,k,\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_{sc}(\eta\nu+\xi\nu^{\perp},k,\nu)e^{i\eta y}d\eta,$$

and obtain

$$\Psi_{sc}(y,\xi,k,\nu) = -\frac{i}{2}\,\frac{k^2 e^{i\sqrt{k^2-\xi^2}\,|y|}}{\sqrt{k^2-\xi^2}} \int \hat{O}(\sqrt{k^2-\xi^2}\,\nu+\xi\nu^{\perp}-p')\Psi(p',k,\nu)dp', \tag{5.6}$$

for $|\xi| < k$. We denote

$$\mu = \frac{1}{k}(\sqrt{k^2-\xi^2}\,\nu+\xi\nu^{\perp}) \quad \text{if } y>0,$$

and

$$\mu = \frac{1}{k}(-\sqrt{k^2-\xi^2}\,\nu+\xi\nu^{\perp}) \quad \text{if } y<0.$$

Here μ is a unit vector. We also have $k|\mu\cdot\nu| = \sqrt{k^2-\xi^2}$ and $\text{sign}(y) = \text{sign}(\mu\cdot\nu)$. Thus,

$$\Psi_{sc}(y,\mu,k,\nu) = -\frac{i}{2}\,\frac{k\,e^{ik\mu\cdot\nu y}}{|\mu\cdot\nu|} \int \hat{O}(k\mu-p')\,\Psi(p',k,\nu)dp'. \tag{5.7}$$

Using Born approximation by setting $\Psi(p',k,\nu) = \delta(p'-k\nu)$ in (5.7) we obtain

$$\hat{O}(k\mu-k\nu) = \frac{2i\,|\mu\cdot\nu|}{k}\,e^{-ik\mu\cdot\nu y}\Psi^{b}_{sc}(y,\mu,k,\nu), \tag{5.8}$$

where Ψ^{b}_{sc} denotes the scattered field in Born approximation.

We can measure the scattered acoustic field outside the ball B_R. Let us fix $|y|>R$ and note that $\text{sign}(y) = \text{sign}(\mu\cdot\nu)$. We make use of the representation in (3.3) (where we set $n = 2$) to obtain

$$O_{LP}(x) = \frac{ik}{4\pi^3} \int_{|\nu|=1} \int_{|\mu|=1} (1-(\mu\cdot\nu)^2)^{1/2} |\mu\cdot\nu| e^{-ik|\mu\cdot\nu|y} \Psi_{sc}^{b}(y,\mu,k,\nu) e^{i(k\nu-k\mu)\cdot x} d\omega_\nu d\omega_\mu \,. \quad (5.9)$$

The formula (5.9) is a backpropagation inversion formula which was first obtained by A. J. Devaney [1] and is presented here in a slightly different form.

The case of Rytov approximation is analogous to Born approximation and can be found in Ref. [1]. We note that in the case of a plane incident wave there is a simple relation between the scattered field in Born and Rytov approximations (see [1], for example), and we can obtain the expression for $\hat{O}(k\mu-k\nu)$ in Rytov approximation using relation in (5.8).

In conclusion let us emphasize that the use of the representation in (3.3) in combination with formulae (4.3), (4.9) and (5.8) allows us to compute contributions of each separate experiment independently. (A separate experiment is a measurement made with a fixed direction of the incident field). Then we integrate over all experiments. Such an integration is a computation of a spherical mean and, thereby, is a stable numerical procedure.

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