SYMPLECTIC MAPS

First used mathematically by Hermann Weyl, the term *symplectic* arises from a Greek word that means "twining or plaiting together." This is apt, as symplectic systems always involve a pair of n-dimensional variables, the configuration q, and momentum p, which are intertwined by the symplectic two form

$$\omega = \mathrm{d}p \wedge \mathrm{d}q. \tag{1}$$

This antisymmetric, bilinear form acts on a pair of tangent vectors and computes the sum of the areas of the parallelograms formed by projecting the vectors onto the planes defined by each canonical pair $(q_i, p_i),\ i = 1, \dots, n$, giving

$$\omega(v, w) = \sum_{i=1}^{n} (v_{p_i} w_{q_i} - v_{q_i} w_{p_i}).$$

A diffeomorphism $f : X \to X$ on a $2n$-dimensional manifold X with coordinates $z = (q, p)$ is symplectic if it preserves the symplectic form, that is, if $f^*\omega = \omega$ (Arnold, 1989; McDuff & Salamon, 1995). If we write $z' = (q', p') = f(q, p)$, the symplectic condition becomes

$$Df^t J Df = J, \quad \text{where} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2}$$

Here $Df_{ij} = \partial f_i / \partial z_j$ is the Jacobian matrix of f, J is the Poisson matrix, and I is the $n \times n$ identity matrix. Equivalently, Stokes' theorem can be used to show that the loop action, $A[\gamma] = \oint_\gamma p\mathrm{d}q$, is preserved by f for any contractible loop γ on X. If f preserves the loop action for all loops, even those that are not contractible, then it is *exact symplectic*.

When $n = 1$, the symplectic condition is equivalent to $\det(Df) = 1$, so that the map is area- and orientation-preserving. Examples include the much studied *standard map* and the area-preserving Hénon quadratic map $f(q, p) = (p + a - q^2, -q)$ (Meiss, 1992). When $n > 1$, the symplectic condition implies volume and orientation preservation, but as we will see, is stronger than this. A generalization of the standard map to higher dimensions is the map

$$\begin{aligned} q' &= q + p - \nabla V(q), \\ p' &= p - \nabla V(q)\ , \end{aligned} \tag{3}$$

where $q \in \mathbb{T}^n$ is an angle, $p \in \mathbb{R}^n$ is its conjugate momentum, and $V(q)$ is a periodic potential. This map is exact symplectic for any V. Beginning in 1972, Claude Froeschlé studied the case $n = 2$ and $V(q) = a \cos q_1 + b \cos q_2 + c \cos(q_1 + q_2)$. Similarly, the natural generalization of the Hénon map is the quadratic symplectic map whose normal form has been given by Moser (1994).

Applications

Symplectic maps arise from Hamiltonian dynamics, because these preserve the loop action. Thus, for example, the time t map of any Hamiltonian flow is symplectic, as is a Poincaré return map defined on a cross section. It is often easier to study the Poincaré map instead of the flow, because the dimension is reduced. Even though explicit construction of the map is typically impossible, approximation methods often suffice.

For example, the time T map of a periodically forced system $H(q, p, t) = H(q, p, t + T)$, such as a pendulum with an oscillating support, is symplectic (*See* **Hamiltonian systems**). An extreme example is $H = \frac{1}{2}p^2 - k\cos(q)\bar{\delta}(t)$, where $\bar{\delta}$ is the periodic Dirac delta function; the corresponding map is the standard map.

As Birkhoff showed, an ideal billiard (a free particle moving inside a rigid, convex table) is naturally written as a symplectic map. The canonical coordinates are the position and the tangential momentum at a collision point. Symplectic maps also arise naturally in systems where the forces are localized in time or space. For example, a circular particle accelerator or storage ring has a sequence of accelerating and focusing elements that can be modeled by a composition of symplectic maps, providing the damping effects of radiation can be neglected (Forest, 1998).

Area-preserving maps also arise in the study of the motion of Lagrangian tracers in incompressible fluids or of particles tightly gyrating around magnetic field lines. In particular, when one component of the field is particularly strong, such as in the plasma device called a tokamak, the transverse dynamics reduces to an area-preserving map.

Autonomous canonical transformations are symplectic maps. For example, if $F(q, q')$ is a generating function for a canonical transformation, then it generates a symplectic map. In particular, the Froeschlé map (3) is generated by $F(q, q') = \frac{1}{2}(q' - q)^2 - V(q)$.

An algorithm that respects the symplectic nature of Hamiltonian dynamics is called a symplectic integrator. A first-order symplectic algorithm with time step Δt for the Hamiltonian $H(q, p)$ is generated by $F(q, p') = qp' + \Delta t H(q, p')$ where $\mathrm{d}F = q'\mathrm{d}p' + p\mathrm{d}q$, giving the map

$$q' = q + \Delta t \frac{\partial H}{\partial p'}(q, p'), \quad p' = p - \Delta t \frac{\partial H}{\partial q}(q, p'). \tag{4}$$

Note that the map is implicit since H is evaluated at p'. However, for the case that $H = K(p) + V(q)$ this becomes a leap-frog Euler scheme, an example of a "splitting" method. Symplectic versions of many standard algorithms—such as Runge–Kutta—can be obtained (Marsden et al., 1996). While there is still some controversy on the utility of symplectic methods versus methods that, for example, conserve energy and other invariants or have variable time-stepping, they are superior for stability properties since they respect the spectral properties of the symplectic group.

The Symplectic Group

The stability of an orbit $\{\ldots z_t, z_{t+1}, \ldots\}$ where $z_{t+1} = f(z_t)$ is governed by the Jacobian matrix of f evaluated along the orbit, $M = \prod_t Df(z_t)$. When f is symplectic, M obeys (2), $M^t JM = J$. The set of all such $2n \times 2n$ matrices form the symplectic group $Sp(2n)$. This group is an $n(2n+1)$-dimensional Lie group, whose Lie algebra is the set of Hamiltonian matrices—matrices of the form JS where S is symmetric. Thus, every near-identity symplectic matrix can be obtained as the exponential of a Hamiltonian matrix and corresponds to the time t-map of a linear Hamiltonian flow. There are symplectic matrices, however, that are not the exponentials of Hamiltonian matrices; for example, $-I$. As a manifold, the symplectic group has a single nontrivial loop (its fundamental group is the integers). The winding number of a loop in the symplectic group is called the Maslov index (McDuff & Salamon, 1995); it is especially important for semi-classical quantization.

If M is a symplectic matrix and λ is an eigenvalue of M with multiplicity k, then so is λ^{-1}. Moreover $\det(M) = 1$, so M is volume and orientation preserving. A consequence of this spectral theorem is that orbits of a symplectic map cannot be asymptotically stable. There are four basic stability types for symplectic maps: an eigenvalue pair (λ, λ^{-1}) is

- *hyperbolic*, if λ is real and larger than one;
- *hyperbolic with reflection*, if λ is real and less than minus one;
- *elliptic*, if $\lambda = e^{2\pi i\omega}$ has magnitude one;
- part of a *Krein quartet* if λ is complex and has magnitude different from one, for then there is a quartet of related eigenvalues $(\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1})$.

Thus, a periodic orbit can be linearly stable only when all of its eigenvalue pairs are elliptic. For this case, the linearized motion corresponds to rotation with n rotation numbers ω_i.

Symplectic Geometry

Every symplectic map is volume- and orientation-preserving, but the group Symp(X) of symplectic diffeomorphisms on X is significantly smaller than that of the volume-preserving ones. This was first shown in 1985 by Gromov in his celebrated "nonsqueezing" (or symplectic camel) theorem. Let $B(r)$ be the closed ball of radius r in $\mathbb{R}^{2n}$ and $C_1(R) = \{(q, p) : q_1^2 + p_1^2 \leq R^2\}$ be a cylinder of radius R whose circular cross section is a symplectic plane. Since the volume of C_1 is infinite, it is easy to construct a volume-preserving map that takes $B(r)$ into $C_1(R)$ regardless of their radii. What Gromov showed is that it is impossible to do this symplectically whenever $r > R$. This is one example of a *symplectic capacity*, and is leading to a theory of symplectic topology (McDuff & Salamon, 1995).

Another focus of this theory is to characterize the number of fixed points of a symplectic map, that is, to generalize the classical Poincaré–Birkhoff theorem for area-preserving maps on an annulus. Arnold conjectured in the 1960s that any Hamiltonian diffeomorphism on a compact manifold X must have at least as many fixed points as a function on X must have critical points. A Hamiltonian map is a symplectic map that can be written as a composition of maps of the form (4). Conley and Zender proved this in 1985 for the case that X is the $2n$-torus: f must have at least $2n + 1$ fixed points (at least 2^{2n} if they are all nondegenerate) (Golé, 2001).

Dynamics

In general, the dynamics of a symplectic map consists of a complicated mixture of regular and chaotic motion (Meiss, 1992). Numerical studies indicate that the chaotic orbits have positive Lyapunov exponents and fill sets of positive measure that are fractal in nature. Regular orbits include periodic and quasi-periodic orbits. The latter densely cover invariant tori whose dimensions range from 1 to n. Near elliptic periodic orbits, the phase space is foliated by a positive-measure cantor set of n-dimensional invariant tori. There are chaotic regions in the *resonant* gaps between the tori, but the chaos becomes exponentially slow and exponentially small close to the periodic orbit. Some of these observations, but not all, can be proved.

The simplest case is that of an integrable symplectic map, which can be written in Birkhoff normal form: $f(\theta, J) = (\theta + \nabla S(J), J)$. Here (θ, J) are angle-action coordinates (each n-dimensional) and $\Omega = \nabla S$ is the rotation vector. Orbits for this system lie on invariant tori; thus the structure is identical to that for integrable Hamiltonian systems.

The Birkhoff normal form is also an asymptotically valid description of the dynamics in the neighborhood of a nonresonant elliptic fixed point, one for which $m \cdot \Omega(0) \neq n$ for any integer vector m and integer n. However, the series for the normal form is not generally convergent. Nevertheless, KAM theory implies that tori with Diophantine rotation vectors do exist near enough to the elliptic point, providing the map is more than C^3 and that the twist, $\det D\Omega(0)$, is nonzero. Each of these tori is also a Lagrangian submanifold (an n-dimensional surface on which the restriction of the symplectic form (1) vanishes). The relative measure of these tori approaches one at the fixed point.

Nevertheless, the stability of a generic, elliptic fixed point is an open question. Arnold showed by example in 1963 that lower-dimensional tori can have unstable manifolds that intersect the stable manifolds of nearby tori and thereby allow nearby trajectories to drift "around" the n-dimensional tori; this phenomenon is called Arnold diffusion (Lochak, 1993). When the map is analytic, the intersection angles become exponentially small in the neighborhood of the fixed point, and the existence of connections becomes a problem in perturbation theory *beyond all orders*.

Aubry–Mather theory gives a nonperturbative generalization of KAM theory for the case of monotone twist maps when $n = 1$. These are symplectic diffeomorphisms on the cylinder $\mathbb{S} \times \mathbb{R}$ (or on the annulus) such that $\partial q'/\partial p \geq c > 0$. For this case, Aubry–Mather theory implies that there exist orbits for all rotation numbers ω. When ω is irrational, these orbits lie on a *Lipschitz graph*, $p = P(q)$, and their iterates are ordered on the graph just as the iterates of the uniform rotation by ω. They are either dense on an invariant circle or an invariant Cantor set (called a *cantorus* when discovered by Percival). These orbits are found using a Lagrangian variational principle, and turn out to be global minima of the action.

Aubry–Mather theory can be partially generalized to higher dimensions, for example to the case of rational rotation vectors, where the orbit is periodic (Golé, 2001). Moreover, Mather (1991) has shown that action-minimizing invariant measures exist for each rotation vector, though they are not necessarily dynamically minimal. The existence of invariant cantor sets with any incommensurate rotation vector can also be proven for symplectic maps near an anti-integrable limit (MacKay & Meiss, 1992). Finally, converse KAM theory, which gives parameter domains where there are no invariant circles for the standard map, implies that, for example, the Froeschlé map has no Lagrangian invariant tori outside a closed ball in the space of its parameters (a, b, c) (MacKay et al., 1989).

JAMES D MEISS

See also **Aubry–Mather theory; Cat map; Chaotic dynamics; Constants of motion and conservation laws; Ergodic theory; Fermi acceleration and Fermi map; Hamiltonian systems; Hénon map; Horseshoes and hyperbolicity in dynamical systems; Lyapunov exponents; Maps; Measures; Melnikov method; Phase space; Standard map**

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