Towards an Understanding of the Break-up of Invariant Tori

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Abstract—Theories describing the existence, destruction and ultimate fate of invariant tori for Hamiltonian systems of $1^1/_2$ or 2 degrees of freedom (or equivalently area preserving mappings) are well established. Similar results for higher dimensional Hamiltonian systems have proved elusive. We discuss several techniques for studying the existence and break-up of invariant tori for $2^1/_2$ degrees of freedom (or 4 dimensional symplectic mappings): the anti-integrable limit, power series expansions of the conjugacy to rotation, and an approximate renormalization operator.

PRELIMINARIES

We will either consider Hamiltonian flows with periodically time dependent $H(x,y,t)$ or symplectic maps $f: (x,y) \to (x',y')$. Here we let $(x,y) \in T^d \times R^d$ where x is the configuration, assumed to be periodic with period 1, and y is the canonical momentum. A map f is symplectic if it preserves the canonical form $dy \wedge dx$ (for a review see ‹Meiss, 1992 #730›). The map could arise as Poincaré map of the flow or from a generating function F defined through

$$dF = y'dx' - ydx \tag{1}$$

A symplectic map f satisfies the *twist* condition if the matrix $\partial x'/\partial y$ is uniformly definite. This implies that $x'(x,y)$ can be globally inverted to obtain $y(x,x')$, and therefore that the transformation $(x,y) \to (x,x')$ is a diffeomorphism. Thus if the map has twist, F can be written as a function of (x,x'). Eq. (1) implies that the map is given implicitly by

$$y' = \frac{\partial F}{\partial x'} \equiv F_2, \quad y = -\frac{\partial F}{\partial x} \equiv F_1 \tag{2}$$

For the generating function, the twist condition is equivalent to $F_{12} = -[\partial x'/\partial y]^{-1}$ being uniformly definite. For the case of zero net flux, $F(x+m,x'+m) = F(x,x')$ for any integer vector m.

Maps of the “standard” form are generated by $F(x,x') = T(x,x') - V(x)$. It is worthwhile to think of T as the kinetic energy (or coupling energy), and V as the periodic potential. The simplest example, for

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which $T = {}^1/_2(x'-x)^2$, can be interpreted as the lowest order symplectic discretization of the oscillator $\varepsilon\ddot{x} = -\nabla V(x)$.

TORI AND FREQUENCIES

We study the *rotational* tori, that is tori homotopic to the trivial torus y = constant. In addition we assume that the tori are graphs, $y = Y(x)$, over the configuration variables x; this is guaranteed by Birkhoff for d=1, but there is no corresponding result for d>1. The frequency vector ω is the average direction that an orbit moves around the torus (assuming that this limit exists). We let $\omega = (1,\nu) \in R^n$, where $n \equiv d+1$, reserving the first slot for the periodic time dependence, and scaling time so that this component is unity. Then ν is the *winding ratio*. Since we care only about its direction (as defined by ν), ω should be viewed as a point in the projective space RP^d. A torus is conjugate to rigid rotation if there is a continuous $X(\theta)$ such that

$$x_t = \theta + \nu t + X(\theta+\nu t) \qquad (3)$$

where $X(\theta)$ is periodic, $X(\theta+m) = X(\theta)$ for all $m\in Z^d$. There is a similar relation for the flow case.

A frequency is *commensurate* if there is a nonzero integer vector m such that $m\cdot\omega = 0$. Such a relation is a *resonance* condition. If there are no resonances for ω then it is *incommensurate*. If there are two independent resonances for ω, then $\omega = p$ where p is integral (remember the length of ω is unimportant). A frequency $\omega=(1,\nu)$ is *Diophantine* if there is a $K\neq 0$ and $\tau \geq d$ such that for all nonzero $m \in Z^n$, $|m\cdot\omega| > K|m|^{-\tau}$. The smallest value of K for which there are infinitely many solutions of $|m\cdot\omega| < K|m|^{-\tau}$ is called the Diophantine constant $C^\tau(\omega)$; equivalently,

$$C^\tau(\omega) = \liminf_{m\to\infty}\left(\|m\|^\tau \left|m\cdot\omega\right|\right) \qquad (4)$$

The theory of simultaneous approximation of frequency vectors is by no means as complete as the continued fraction theory for d=1, see e.g. ‹Cassels, 1965 #737›. We adopt a Farey approximation technique proposed by Kim and Ostlund ‹Kim, 1986 #593›. Consider d = 2, and begin with the three resonances $m_1 = (1,0,0)$, $m_2 = (0,1,0)$, $m_3 = (0,0,1)$. Each resonance corresponds to a plane in R^3 or a line in RP^2; the set of three resonances delineates a cone (the positive octant) or triangle. The intersection of each pair of resonances defines rational frequencies $p_1= [1,0,0]$, $p_2 = [0,1,0]$, $p_3 = [0,0,1]$. The frequencies p_i also delineate the cone; it is the convex hull of the three vectors. We denote the cone by either of the matrices $M = (m_1,m_2,m_3)^t$ or $P = (p_1,p_2,p_3)$. We assume ω is inside the cone, i.e. $\omega_i \geq 0$.

To construct the Farey sequence for ω, divide the cone using the new frequency $p'= p_1+p_2$, and corresponding resonance $m' = m_1-m_2$. There is now a right and a left cone $P_R = (p_3,p_1,p')$ and $P_L = (p_2,p_3,p')$, or $M_R = (m_3,m',m_2)^t$ and $M_L = (-m',m_3,m_1)^t$. Choose the new cone that contains ω and repeat this transformation, dividing this new cone into two. This gives a sequence of cones that each contain ω. The operations can be represented by the linear transformations $M_S = S^{-1}M$ or $P_S = PS$. Here

$$S = \begin{cases} R & \text{if } (m_1 - m_2)\cdot\omega > 0 \\ L & \text{if } (m_1 - m_2)\cdot\omega \le 0 \end{cases}, \quad R = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5}$$

Note that det(R) = det(L) = 1, $M_S P_S = I$, and $\det(M_S) = \det(P_S) = 1$. Repeating this transformation provides a unique string of matrices $S_i \in \{R,L\}$ for any ω, so that we can think of ω as the sequence $S_1S_2S_3$... It is not difficult to show that if ω is an integer vector (with no common factors) then this sequence eventually terminates when $p_3 = \omega$ ‹Baesens, 1991 #734›.

From the Farey point of view, the simplest incommensurate frequency vectors have periodic Farey sequences. When the period is q, ω is the eigenvector with largest eigenvalue of the nonnegative matrix $S_1...S_q$. This implies that the components of ω are elements of a cubic field: they satisfy $\omega_i = i + j\lambda + k\lambda^2$, where (i,j,k) are integers and λ is the eigenvalue—it satisfies a cubic equation with integer coefficients. The simplest of these $(1,\sigma^2,\sigma)$ which is the eigenvector of L, where σ is the *spiral mean* (coined by Cvitanovic).

$$\sigma^3 = \sigma + 1, \quad \sigma \approx 1.324717957 \tag{6}$$

This vector is an integral basis for the cubic field generated by σ. Similarly the eigenvector of R is $(\sigma^2,1,\sigma)$.

Another interesting cubic vector is $(1,\tau,\tau^2)$, where $\tau = 2\cos(2\pi/7)$, a solution of $\tau^3-\tau^2-2\tau-1=0$. This vector is an integral basis for the totally real cubic field with minimum discriminant (Δ = 43). It has the Farey sequence $L(LRLLRLR)^\infty$. Cusick has conjectured that there are integral bases in the field generated by τ whose Diophantine constants limit on $C^2(\omega) = 2/7$ ‹Cusick, 1974 #805›. It is known that the largest possible Diophantine constant for the three frequency case is at least 2/7, and numerical evidence indicates that this is indeed the supremum ‹Szerkeres, 1985 #934›. Thus τ might be the correct generalization of the golden mean—giving the most robust tori.

Complex Analytic Tori

One technique for determining whether a symplectic map has a rotational torus with a given frequency is to attempt to obtain the conjugacy function by perturbation series. If the map is analytic, then it is reasonable to search for an analytic conjugacy; indeed the analytic KAM theorem implies that the conjugacy is analytic for sufficiently small perturbations from integrability providing the winding ratio is Diophantine. Thus we assume that there is a convergent Fourier series

$$X(\theta) = \sum_{n \in Z^d} X_n e^{2\pi i n\cdot\theta} \tag{7}$$

Substitution of this series, together with the conjugacy relation (3) into the map gives a set of equations for the Fourier coefficients.

There are two cases in which these equations can be dealt with relatively easily. In the first, suppose that F has the form $F(x,x') = {}^1/_2(x'-x)^2 - \varepsilon V(x)$ where V has only finitely many Fourier components. Then we expand the conjugacy as power series in ε. The number of Fourier components at each order in ε is finite ‹Greene, 1981 #187; de la Llave, 1992 #827›. One then finds the values of ε for which this series ceases to converge, thus obtaining a lower bound for the domain of existence of the torus.

A more specialized case is that of a complex map for which the periodic terms contain only one sign of complex exponential—following Percival we use the terminology "semi-" for such maps. As an example ‹Bollt, 1993 #795›, consider the "semi-Froeshlé" map generated by the complex generating function

$$F(x,x') = \frac{1}{2}(x'-x)^2 + \frac{1}{4\pi^2}\left(ae^{2\pi i x_1} + be^{2\pi i x_2} + ce^{2\pi i(x_1+x_2)}\right)$$

The major simplification for semi-maps, is that one can find invariant tori that are analytic functions of the d complex variables $e^{2\pi i\theta_\kappa}$—no negative Fourier components occur. In fact for the semi-Froeshlé map, two of the parameters, a and b, can be absorbed in the definition of a pair of complex variables u $\equiv(ae^{2\pi i\theta_1}, be^{2\pi i\theta_2}) \in C^2$. Thus we rewrite (7) as

$$X(\theta) = -\frac{i}{2\pi}\sum_{n=0}^{\infty} b_n u^n \tag{8}$$

Upon substitution of this expansion into the Lagrangian form of the semi-Froeshlé map we obtain recursion relations for each fixed value of $k = \varepsilon/ab$, the coupling coefficient that involve the small denominator $D_n \equiv \sin^2(\pi n\cdot\nu)$. For Diophantine frequencies D_n is bounded away from zero as $D_n > K|n|^{-4}$.

Upon solving for b_n, we find the domain of convergence of the double series (8). Any power series

$$S = \sum_{n\in N^d} b_n z^n$$

in d-complex variables has a domain of convergence that is a log-convex, complete Reinhardt domain. A *Reinhardt domain* D is a set in C^d for which if $z \in D$ then so is $ze^{i\theta} = (z_1e^{i\theta_1},...)$ for any real phases θ_i. Thus it is conveniently pictured in terms of the magnitudes $|z_i|$. The domain D is *complete* if for each $z \in D$, we have $z' \in D$ whenever $|z_j'| \le |z_j|$, and it is *log-convex* if the set $\log(|D|) \equiv \{(\log(|z_1|), \log(|z_2|),...): z \in D \text{ and } z_j \neq 0\}$ is a convex subset of R^d.

There are several differences with the d = 1 case. In particular the domain of convergence D in C is always a disk, and if $z \notin \bar{D}$ the series diverges. Furthermore a power series in C converges iff it converges absolutely. These facts are no longer necessarily true in C^d, as is illustrated by the series $S = \sum_{j=0}^{\infty}(z_1+z_2)^j$. For this particular ordering of the terms we have the sum $S = (1-z_1-z_2)^{-1}$ for $|z_1+z_2| < 1$. This appears to contradict our theorem, since this domain is not a Reinhardt domain. However by formally rearranging terms, we obtain

$$S \Rightarrow S' = \sum_{n,k=0}^{\infty} C_n^{k+n} z_1^n z_2^k$$

where C is the binomial coefficient. The test for S′ to converge <u>absolutely</u> is

$$\left|S'\right| \le \sum_{n,k=0}^{\infty} C_n^{k+n} r_1^n r_2^k = \sum_{j=0}^{\infty} (r_1+r_2)^j, \quad r_1+r_2 < 1$$

So S' converges absolutely in a subset of the domain for which the original ordering converges. The domain of convergence is $|z_1|+|z_2| < 1$, because the double power series only "makes sense" if it converges for any ordering of its terms (and then it converges absolutely). As a final example consider the series

$$S = \sum_{j=0}^{\infty} z_1{}^j z_2 = \frac{z_2}{1-z_1}$$

This series converges in $G = \{z : |z_1| <1\} \cup \{z : z_2 =0\}$. The domain of convergence is the interior of G, which is $D =\{z : |z_1| <1\}$. Thus S does not automatically diverge on the complement of the closure of D.

We do not have a sophisticated technique for determining ∂D. Our simple idea is to reduce the d-dimensional series to sequence of one dimensional ones. First we use absolute convergence to focus on the real series: $S' = \sum_{n \in N^d} \left|b_n\right| r^n$. One way to convert this to a single series is to fix all but one of the radii, and "do" the interior sums, leaving the one dimensional sum $S' = \sum_{n_1=0}^{\infty} B(r_2 \ldots r_d) r_1^n$. However, this can not be implemented numerically, since the interior sums are infinite. A better idea is to fix a set of slopes $s_i \equiv r_i/r_1$, i=2,...d, and reorder the series using these variables.

$$S' = \sum_{m=0}^{\infty} B_m(s_2,s_3,\ldots s_d) r_1^m, \quad B_m \equiv \sum_{n_2=0}^{m} s_2^{n_2} \sum_{n_3=0}^{m-n_2} s_3^{n_3} \ldots \sum_{n_d=0}^{m-n_1-\ldots n_{d-1}} \left|b_{m-\Sigma n_k, n_2,\ldots,n_d}\right| s^{n_d} \tag{9}$$

Now each of the interior sums is finite and B_m can be computed "exactly". The radius estimated by the asymptotic growth rate

$$\log\left(r_1(s_2,\ldots s_d)\right) = -\limsup_{m\to\infty} \frac{1}{m} \log\left(B_m(s)\right) \tag{10}$$

Completeness of D implies that $r_1(s)$ is a function. In practice we use this definition when the $s_i < 1$ so that the terms decrease, permuting the definition to retain this in the other wedges.

Returning now to the semi-Froeshlé map: the radii $(|u_1|,|u_2|) = (|a|,|b|)$ in fact represent the parameter values at which the analytic torus is destroyed. Thus we obtain, for each k, the graph of the boundary of D in (a,b). Therefore the domain of convergence of the series for an analytic torus cannot, for example, have holes or disconnected components—like the domain of existence of a invariant circle for the multi-harmonic standard map ‹Wilbrink, 1990 #725›. Furthermore, for each k, the domain must be convex when plotted in the coordinates log(a) vs log(b). When $k = \varepsilon = 0$, the domain is simply the rectangle $|a| < a^{ss}(\omega_1)$, $|b| < a^{ss}(\omega_2)$ where $a^{ss}(\omega)$ is the critical parameter for the semi-standard map, $\delta^2 x = \frac{ia}{2\pi} e^{2\pi i x}$.

In ‹Bollt, 1993 #795› we studied the domain of convergence of the series for several algebraic winding ratios, the spiral mean (σ,σ^2) and the quartic vectors $(\gamma,\sqrt{2})$ and $(\gamma,(3-\sqrt{2})/7)$. Subsequently, we also computed the domain for the cubic (τ,τ^2); it appears very similar to the other cases. What surprised me about these calculations is that the domain boundaries appear to be smooth as far as we can tell. I would have expected something like the fractal set of cusps observed for the two-harmonic standard map ‹Ketoja, 1989 #572›; however, log-convexity would provide a strong constraint on this, so perhaps we should expect the cusps to occur only for some derivative of the boundary. There are several interesting unexplained phenomena. The domain of convergence is defined to be the intersection of the domains for the two components of X(u); these are not the same. Based on what happens for the area preserving case, we might expect that there is a remnant cantor set when the series does not converge, and that when only one of the two series converges, there is a cantor set of circles.

APPROXIMATE RENORMALIZATION

The approximate renormalization theory introduced by Escande and Doveil ‹Escande, 1981 #147› has proved extremely useful in understanding the breakup of tori for $d = 1$. We extend their results by studying the natural generalization of their system, a particle in the plane acted on by three travelling waves ‹MacKay, 1994 #927›. This Hamiltonian can be rescaled to take the form

$$\begin{aligned} H &= \tfrac{1}{2}(u,v)\cdot\begin{pmatrix}\alpha & \beta \\ \beta & \gamma\end{pmatrix}\cdot\begin{pmatrix}u \\ v\end{pmatrix} + V(x,y,z) \\ V &= A\cos(2\pi x) + B\cos(2\pi k y) + C\cos(2\pi \ell z) \\ z &= t - x - y \end{aligned} \tag{11}$$

One unusual aspect of H is the nondiagonal mass matrix—it is important to keep this as it will be generated in any case by the Farey approximation.Without loss of generality, the wavenumbers (k,ℓ) can be taken to be positive and the mass matrix has unit determinant, $\alpha\gamma-\beta^2 = 1$. Thus this system has seven parameters.

The Hamiltonian is periodic with periods $(1,1/k,1/\ell)$ in the configuration variables (x,y,z)—so the configuration space can be taken to be the three torus $T^3 = \{x \bmod 1,\ y \bmod 1/k,\ z \bmod 1/\ell\}$. When A=B=C=0, the momenta (u,v) are constant in time and every orbit lies on a three torus. If $\omega(u,v)$ is incommensurate, the orbit densely covers the torus. If ω is Diophantine, then the KAM theorem implies that there is a torus with this frequency for small values of the amplitudes. We are interested in determining the parameters for which such a torus is destroyed.

The technique is to perform a succession of canonical transformations to coordinates that are more closely aligned with the incommensurate flow. Our renormalization is a coordinate transformation that focuses on a region of phase space in which orbits of a given frequency ratio are expected. We can define two such transformations corresponding to the L and R Farey steps. We define a canonical transformation to eliminate one of the resonances and then transform the new Hamiltonian back to its original form.

Suppose formally that each of the parameters $A,B,C = O(\varepsilon)$. We begin by eliminating the $m_2 = (010)$ resonance by a near identity canonical transformation. Then, for the “L” transformation, define the new coordinates on T^3

$$\begin{pmatrix} x' \\ k'y' \\ \ell' z' \end{pmatrix} = L^{-1}\begin{pmatrix} x \\ ky \\ \ell z \end{pmatrix} + O(\varepsilon) \tag{12}$$

In order to maintain the form $z' = t' - x' - y'$ the new wavenumbers must be

$$k' = \frac{\ell}{k}, \quad \ell' = \frac{1}{1+k} \tag{13}$$

Upon defining new momenta corresponding to these coordinates, scaling time to $t' = kt$ and scaling the momenta to normalize the mass matrix, the Hamiltonian has the same form as (1) to $O(\varepsilon^3)$ if we identify the new parameters

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = \frac{1}{1+k}\begin{pmatrix} 1/k & -2 & k \\ 1 & 1-k & -k \\ k & 2k & k \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$
$$A' = \frac{(1+k)^3\beta}{2k^2} AB, \quad B' = \frac{1+k}{k} C, \quad C' = \frac{1+k}{k} A \tag{14}$$

This is the approximate renormalization operator. There is a corresponding operator for the right Farey step.

The simplest frequency vectors under renormalization correspond to the fixed point of L; the frequency vector is then the spiral mean (6). The wavenumber renormalization (13) is decoupled from the parameters, so we can consider it separately. The wavenumber map is contracting and has the unique real fixed point $(k,\ell) = (\sigma^{-1},\sigma^{-2})$ where σ is the spiral mean. The fixed point is a spiral focus. The mass renormalization is a linear map, and has been constructed to preserve the subspace $\alpha\gamma - \beta^2 = 1$. Since the wavenumber map is contracting, we can evaluate the mass map at the fixed point $k = \sigma^{-1}$. This gives the eigenvalues

$$\lambda_1 = 1, \quad \lambda_{2,3} = e^{\pm i\psi}$$

where $\cos(\psi) = 0.5(\sigma - 1)$. Therefore this map is <u>not</u> contracting— in general the mass matrix rotates with a rotation number $\psi/2\pi \approx 2/9$. This violates the notion of "universality": asymptotics of the the orbit under the renormalization depend on the parameters of the initial Hamiltonian.

The parameter map depends on the wavenumber k and the mass matrix through β. Consider first the case when $\beta = \cot(\psi)$ is fixed. In this case there are two fixed points, $A = B = C = 0$, the KAM fixed point, and the critical fixed point with all three parameters nonzero. The KAM fixed point is stable. The linearization about critical point shows that this point is a spiral saddle with characteristic polynomial $\lambda^3 - \lambda^2 - 1 = 0$ (interestingly, this polynomial is not related to the spiral mean), so that

$$\lambda_1 = \delta \approx 1.465571232, \quad |\lambda_{2,3}| = \delta^{-1/2} \tag{15}$$

Thus there is a one dimensional unstable manifold, and a two dimensional, spiral stable manifold. The contraction on the stable manifold is rather slow.

For the general case, β is not fixed, and the amplitude map is periodically forced. However, there is still a codimension one center-stable manifold which has a one dimensional unstable manifold. On the center manifold the parameters converge to a circle on which the dynamics is a simple rotation with rotation number $\psi/2\pi$. The rotations arise because successive rational approximants of the incommensurate vector spiral inwards (the analogous oscillation in $1^1/_2$ degrees of freedom is responsible for the momentum scaling eigenvalue being negative).

Thus, if we take our model at face value, it predicts that a typical one parameter system is not "self-similar" at criticality. Instead properties of the system such as the stability parameters of periodic orbits (i.e. the residues) are predicted to oscillate with rotation number of approximately 2/9. The amplitude of the oscillation will depend upon the system studied; in our model it depends upon the off-diagonal element in the mass matrix (alternatively one can think of this as coming from the wavevectors not being perpendicular).

Indeed, previous attempts to find the critical point for a spiral mean torus have seen evidence for these oscillations. Artuso et al ‹Artuso, 1991› studied a 3D volume preserving map, and found that the residues of periodic approximations to a spiral mean torus oscillated, apparently with period 9.

Now the true renormalization dynamics need not look the same as our approximate model, even if it is very good approximation. This is because no rotation is stable to perturbation. Arbitrarily small perturbations of a rotation can make the fixed point weakly attracting or repelling, and can generate weakly attracting or repelling invariant circles around the fixed point, or chains of periodic orbits, or Birkhoff attractors or worse! What *is* stable to perturbation, however, is the fixed point and a 2D normally hyperbolic invariant manifold containing it, with 1 unstable normal direction, the remaining normal directions being attracting. The stable manifold of this 2D normally hyperbolic manifold has codimension 1 and can be expected to be the boundary of KAM theory. It would be worth trying to find the fixed point, because it would be an important handle on the normally hyperbolic manifold. We call it a codimension-3 fixed point because in our model it has three eigenvalues which are not strictly inside the unit circle, so it has three-dimensional center-unstable manifold, and hence requires three parameters to find it.

ANTI-INTEGRABILITY

A major unresolved question concerns the fate of a torus upon destruction: does it become a cantorus as happens for d = 1 according to Aubry-Mather theory? There is no generalization of this result to the higher dimensional case. We can show, however that many maps have cantori for every frequency when they are sufficiently far from integrability—i.e. near the anti-integrable limit ‹MacKay, 1992 #687›.

For a twist map, the *action* $W_{j,k}[x]$ of a sequence $[x] = \{x_j,x_{j+1},\ldots,x_k\} \in R^{d(k-j+1)}$ is defined as $W_{jk}[x] = \sum_{t=j}^{k-1} F(x_t,x_{t+1})$. It is easy to see that a sequence with fixed endpoints is an orbit segment iff it is a critical point of W: $\partial W/\partial x_t = 0$ for k<t<j. We have no difficulty in extending this notion to infinite sequences $[x] \in R^{dZ}$, except that the action itself is formally infinite. The orbit in phase space is

completely defined by the configuration through $y_t= F_2(x_{t-1},x_t)$. A sequence is a periodic orbit with frequency vector $(p,q) \in Z^{d+1}$ if $x_n= x_0+p$, and it is a stationary point of $W_{0,n}$ for $0\le t<n$.

Consider a generating function of the form

$$F(x,x') = \varepsilon T(x,x') - V(x)$$

From the variational point of view, the map generated by F corresponds to a linear chain of particles at points x_j coupled by harmonic springs in a periodic potential V. We call the case $\varepsilon=0$ the *anti-integrable* limit. This is a singular limit because the generating function no longer satisfies the twist condition, and thus does not actually generate a map through (2). However, one can still consider, for a sequence [x], the critical points of W. In the anti-integrable limit, these sequences are simply arbitrary sequences of critical points of V(x). Suppose that the set of such critical points of V is $C = \{c^i\}$. Then a valid “orbit” is any choice $x_t \in C$. The interpretation is that the spring constants are zero, so the particles sit at points of zero force, $\nabla V=0$.

Continuation from the anti-integrable limit is much easier than from the integrable limit. The beautiful result is that every nondegenerate sequence with bounded acceleration can be continued to nonzero ε. An anti-integrable configuration is nondegenerate if each of the points x_t is a nondegenerate critical point of V. The *acceleration* of a sequence is defined as

$$a[x] = \sup_t \left| T_2(x_{t-1},x_t) + T_1(x_t,x_{t+1}) \right|$$

since for the standard kinetic energy, a[x] is the supremum of the second differences $|x_{t-1} - 2x_t + x_{t+1}|$, the discrete version of acceleration. Orbits with bounded acceleration have momentum changes that are bounded.

> **Theorem** (MacKay and Meiss): Given $A > 0$, there is $\varepsilon_0(A) > 0$ such that all nondegenerate anti-integrable sequences with $a[x] \le A$ persist for $\varepsilon < \varepsilon_0$, and remain nondegenerate.

This is a simple consequence of the implicit function theorem.

As a corollary to the theorem, we can construct cantori for any incommensurate vector v. The anti-integrable limit of a cantorus is defined through a function X, as in (3), but now we allow X to be discontinuous, setting $X(\theta) = [\theta]$, where [] is the map from R^d to the critical point in the same fundamental domain as its argument. These states have bounded acceleration, and it is easy to show that they continue to cantor sets. Furthermore, these cantori are hyperbolic. One can continue these orbits until they loose hyperbolicity; however, it is not known whether a cantorus becomes a torus at this point.

CONCLUSIONS

We have seen that some of the results from the theory of area preserving twist maps can be generalized to higher dimensions. Certainly KAM theory tells us that near enough to the integrable case, there are tori with every Diophantine frequency. Similarly the anti-integrable theory implies that there are cantori for every incommensurate frequency near the anti-integrable limit. The converse KAM theory ‹Mather, 1984 #297; MacKay, 1985 #283› generalizes to a more limited extent. For d = 1, one can show that there are parameter domains outside of which there are no rotational invariant circles. This is based on a theorem of Birkhoff that every such circle is the graph of a Lipschitz function. For d>1 the analogue of this theorem is not known. If one assumes that invariant tori for twist maps are Lagrangian graphs, then the converse KAM theory applies ‹MacKay, 1989 #525›. Numerical studies do not contradict this assumption.

Our approximate renormalization gives a possible description for the boundary of existence of the spiral mean torus. The boundary is a codimension one surface in the space of parameters that is the center-stable manifold of a critical fixed point of the renormalization operator with the single unstable eigenvalue $\delta \approx 1.4655$ and two neutral eigenvalues. All orbits on the center-stable manifold are attracted to the center manifold under renormalization. The renormalization dynamics on the center manifold is a rotation with irrational winding ratio.

One might try to use analogue of Greene’s residue criterion to find this fixed point by studying the stability of the periodic orbits making up successive cones in the Farey sequence for ω. Each of the three orbits bounding the cone has a pair of residues. One must find a set of parameter values for which the limit of all of these residues neither goes to infinity nor zero. This is difficult numerically because long orbits must be obtained in order to see if there is any asymptotic behavior at all.

We conjecture that the breakup boundary may have various components corresponding to the direct formation of full cantorus which we know exists close enough to the anti-integrable limit or of a partial cantorus corresponding to a Cantor set cross a circle with various homotopy types. It is also possible that some parameter directions correspond to the formation of a Sierpinski set or even a more exotic topology. It is reasonable that the codimension three fixed point will form the organizing center for these bifurcations.

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