Background

- Ever since Stokes' famous 120° cusp conjecture, there has been great interest in traveling wave solutions of the water wave equations.
- In 1972, Fenton used complex-variable techniques to study Stokes' wave of maximum height.
- To do so, he obtained a series

$$\eta(x) = \eta_1(x) + \epsilon\eta_2(x) + \cdots + \epsilon^9\eta_9(x) + O(\epsilon^{10})$$

where $\eta_j = \sum_{i=1}^{j} A_{ij}\mathrm{sech}^{2i}(x)$.

- Question: Can one do something similar to study traveling water waves in (2+1) dimensions (with surface tension)?
- Is there a wave of maximum depth in (2+1) dimensions?

- Main obstacle: Can't use complex variables in (2+1) dimensions.
- Solution: Use the reformulation of full water waves given by Ablowitz, Fokas, and Musslimani (2006) in terms of η and $q \equiv \varphi(x, \eta)$.
- Reformulation fixes the domain.
- Reformulation involves Bernoulli's equation, and a new nonlocal equation of Fourier type.

Non-dimensionalized AFM System

- Nonlocal equation, valid for all $k = (k_1, k_2) \in \mathbf{R}^2$

$$\int_{\mathbf{R}^2} e^{ik\cdot x} \cosh(|k|(\epsilon\eta + 1))\eta_t \, dx =$$
$$\frac{i}{\mu} \int_{\mathbf{R}^2} e^{ik\cdot x} \frac{\sinh(|k|(\epsilon\eta+1))}{|k|} (k \cdot \nabla) \, q \, dx$$

where $|k| \equiv \sqrt{k_1^2 + \gamma^2 k_2^2}$

- Bernoulli's equation for $q(x, y, t) \equiv \varphi(x, y, \eta(x, y, t), t)$

$$q_t - \frac{1}{2}|\nabla q|^2 + g\eta - \frac{(\eta_t + \nabla q \cdot \nabla\eta)^2}{2\,(1 + |\nabla\eta|^2)} = \frac{\sigma}{\rho'}\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}} \right) \tag{1}$$

- Want to study traveling wave solutions of AFM system
- Let $X = x - c_X t$ and $Y = y - c_Y t$.
- Take $\mu^2 = \gamma^2 = \epsilon$
- One may express AFM system in terms of variables (X, Y). For example, nonlocal equation becomes:

$$
-\int_{\mathbf{R}^2} e^{ikX} \cosh\left(\sqrt{k_1^2 + \epsilon^2 k_2^2}\,(\epsilon\eta + 1)\right)(c_X \eta_X + c_Y \eta_Y)\; dx =
$$

$$
\frac{i}{\sqrt{\epsilon}} \int_{\mathbf{R}^2} e^{ik\cdot X} \frac{\sinh\left(\sqrt{k_1^2 + \epsilon^2 k_2^2}\,(\epsilon\eta + 1)\right)}{\sqrt{k_1^2 + \epsilon^2 k_2^2}} (k \cdot \nabla)\, q\, dx
$$

- Want: Asymptotic series

$$q \sim q_0 + \epsilon q_1 + \epsilon^2 q_2 + \dots$$
$$\eta \sim \eta_0 + \epsilon \eta_1 + \epsilon \eta_2 + \dots$$

- Strategy: Use Poincare-Stokes method.
- Let $c_X = 1 + \epsilon c_{X_1} + \epsilon^2 c_{X_2} + \dots$ and $c_Y = \epsilon c_{Y_1} + \epsilon^2 c_{Y_2} + \dots$ where the c_{X_j} and c_{Y_j} are unknowns to be solved for.
- From AFM system, can use Mathematica to derive equations for q_j:

$$\mathrm{KP}(q_0) = 0$$
$$Lq_j = F_j(q_0, \dots, q_{j-1}), j = 1, 2, \dots,$$

  where L is a linear fourth-order PDE with non-constant coefficients
- Once have q, also have $\eta = \eta_0(q) + \epsilon \eta_1(q) + \dots$

To obtain main equations

$$KP(q_0) = 0$$
$$Lq_j = F_j(q_0, \ldots, q_{j-1}), j = 1, 2, \ldots$$

for q_0, q_1, and q_2 (say) we do following:

- Expand Bernoulli equation in ϵ, neglecting terms of order $O(\epsilon^3)$
- Use above to express η in terms of q: $\eta = \eta_0(q) + \epsilon\eta_1(q) + \epsilon^2\eta_2(q) + O(\epsilon^3)$
- Expand the nonlocal equation in ϵ, neglecting terms of order $O(\epsilon^3)$, and take Fourier transform of result
- Put in $\eta = \eta_0(q) + \epsilon\eta_1(q) + \epsilon^2\eta_2(q) + O(\epsilon^3)$ into expanded nonlocal equation to get equation for q only
- Finally, use ansatz $q \sim q_0 + \epsilon q_1 + \epsilon^2 q_2 + O(\epsilon^3)$ for q to get final equations for q_j

- In 1-D, equations reduce to

$$\begin{aligned} \mathrm{KdV}(\eta_0) &= 0 \\ Lq_j &= F_j(\eta_0, \dots, \eta_{j-1}), j = 1, 2, \dots \end{aligned}$$

- One may solve these equations analytically for $\eta_j = \sum_{i=1}^{j} A_{ij}\mathrm{sech}^{(2i)}(x)$ and for the speeds c_j in terms of an arbitrary parameter α.
- By normalizing appropriately and taking $\sigma = 0$, we checked that our series for η and speed c (in terms of ϵ) agree with Fenton's
- In 2-D, don't know how to find analytic solutions to equations
- Therefore, must
    1. find a way to determine speeds c_{Y_j} and c_{X_j}, $j = 1, 2, \dots$
    2. once determine speeds, solve for q_j's iteratively using numerics

- Recall that want to solve

$$
\begin{aligned}
\mathrm{KP}(q_0) &= 0 \\
Lq_j &= F_j(q_0, \ldots, q_{j-1}), j = 1, 2, \ldots
\end{aligned}
$$

- $KP(q_0) = 0$ satisfied by lump solution

$$
-\frac{32\alpha_2^2\left(X\left(\sigma-\frac{1}{3}\right)^{-\frac{1}{4}}-2\sqrt{3}Y\alpha_1\right)\left(\sigma-\frac{1}{3}\right)^{3/4}}{48Y^2\alpha_2^4+4\left(X\left(\sigma-\frac{1}{3}\right)^{-\frac{1}{4}}-2\sqrt{3}Y\alpha_1\right)^2+1}
$$

  if speeds c_{Y_0} and c_{X_0} chosen correctly as functions of α_1 and α_2
- By assuming that $q \sim$ lump solution as $X^2 + Y^2 \to \infty$, can get $c_{X_j} = c_{X_j}(\alpha_1, \alpha_2)$ and $c_{Y_j} = c_{Y_j}(\alpha_1, \alpha_2)$.
- Now have three free parameters: α_1, α_2, and ϵ. Correspond to μ, γ and ϵ of original non-dimensionalization.
- All that remains: use numerics to solve sequence of PDEs iteratively.

To solve PDEs $Lq_1 = F_1(q_0)$ and $Lq_2 = F_2(q_0, q_1)$, do:

- Fix α_1 and α_2
- Truncate domain to $(-a, a) \times (-a, a)$ and use 6th order equispaced FD stencil with periodic boundary conditions
- Once have approximate grid values for q_1, approximate derivatives $q^{(m,n)}(X, Y)$ that occur in $F_2(q_0, q_1)$ as follows:
  1. Take (modified) FFT of grid data that approximate function q_1
  2. Multiply result pointwise by matrices $(-i\omega_X)^m(-i\omega_Y)^n$, where $\omega_{X_{pq}} = -\pi/a(-N/2 + p)$ and $\omega_{Y_{pq}} = -\pi/a(-N/2 + q)$
  3. Take (modified) inverse FFT of result
- Use this to approximate $F_2(q_0, q_1)$, and solve $Lq_1 = F_1(q_0)$ for q_2

- So, when we fix values for α_1 and α_2, we
    1. Have values for the speeds c_X and c_Y (to order $O(\epsilon^2)$)
    2. Have potential $q = q_0 + \epsilon q_1 + \epsilon^2 q_2$, valid to order $O(\epsilon^2)$
    3. We then can get $\eta = \eta_0 + \epsilon_1 + \epsilon^2 \eta_2$
    4. Choose ϵ such that $\max |\eta_0 + \epsilon_1 + \epsilon^2 \eta_2| = 1$
- With above choice, ϵ is the non-dimensional maximum amplitude of η.
- As α_1 and α_2 vary, we can get the speed/amplitude relationship for water waves (to order $O(\epsilon^2)$)
- Question to be answered: In above normalization, as ϵ goes to 1, dimensional η approaches bottom boundary. What happens to numerical solution? Can there be a lump-type solution that gets arbitrarily close to the bottom?