Dynamics and stability of localized nonlinear waves in inhomogeneous media

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Collaborators

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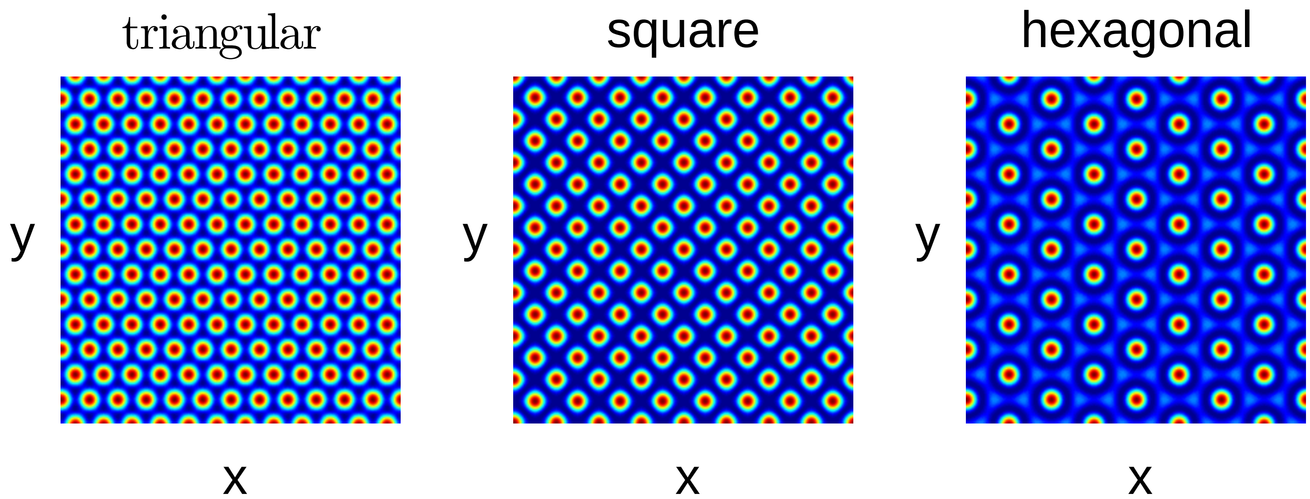
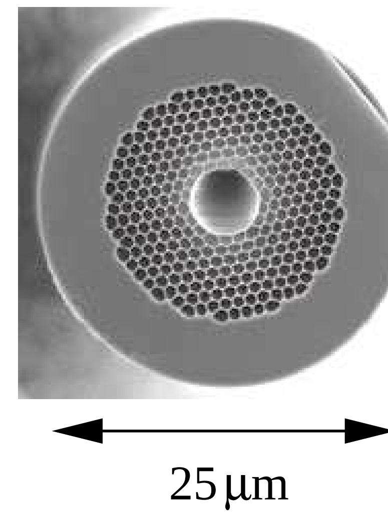
- Nalan Antar and İlkay Bakırtaş, Istanbul Technical University
- Gadi Fibich and Yonatan Sivan, Tel Aviv University
- Rafael Piestun and Ethan Schonbrun, CU Electrical Engineering
- Michael Weinstein, Columbia University

publication:
Ablowitz, Ilan, Schonbrun, and Piestun,
Phys. Rev. E - Rapid. Comm., Sep. ’06

much work in progress...

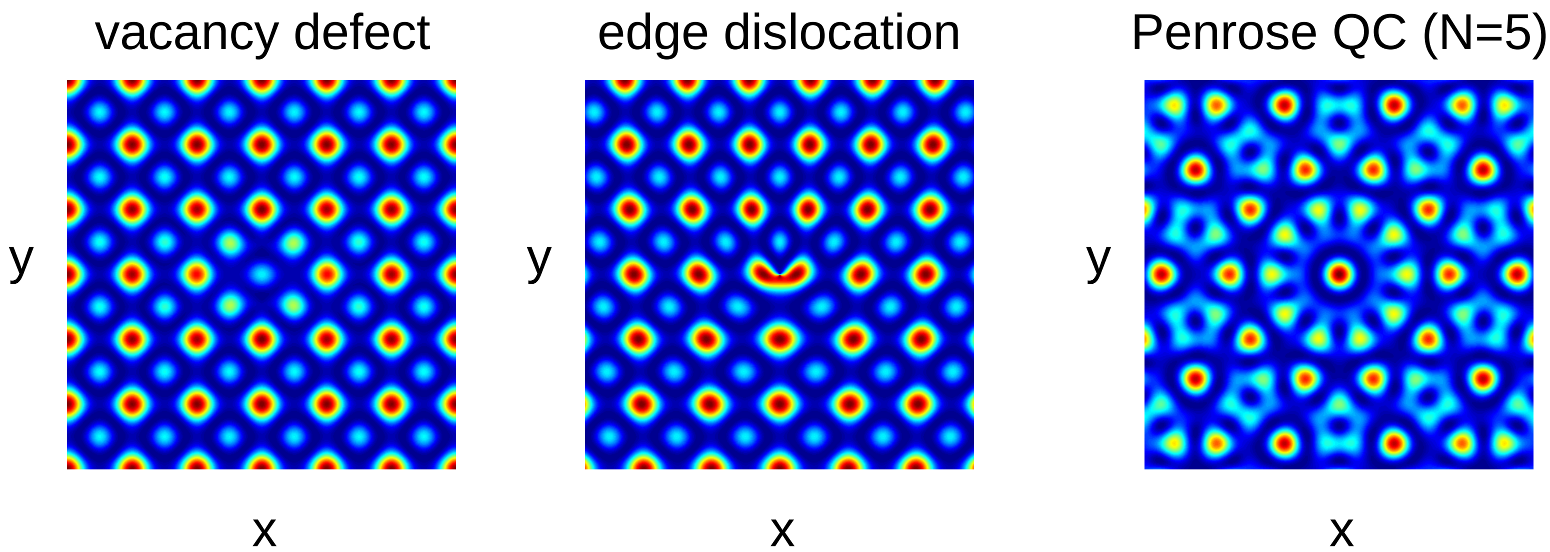
Lattice systems





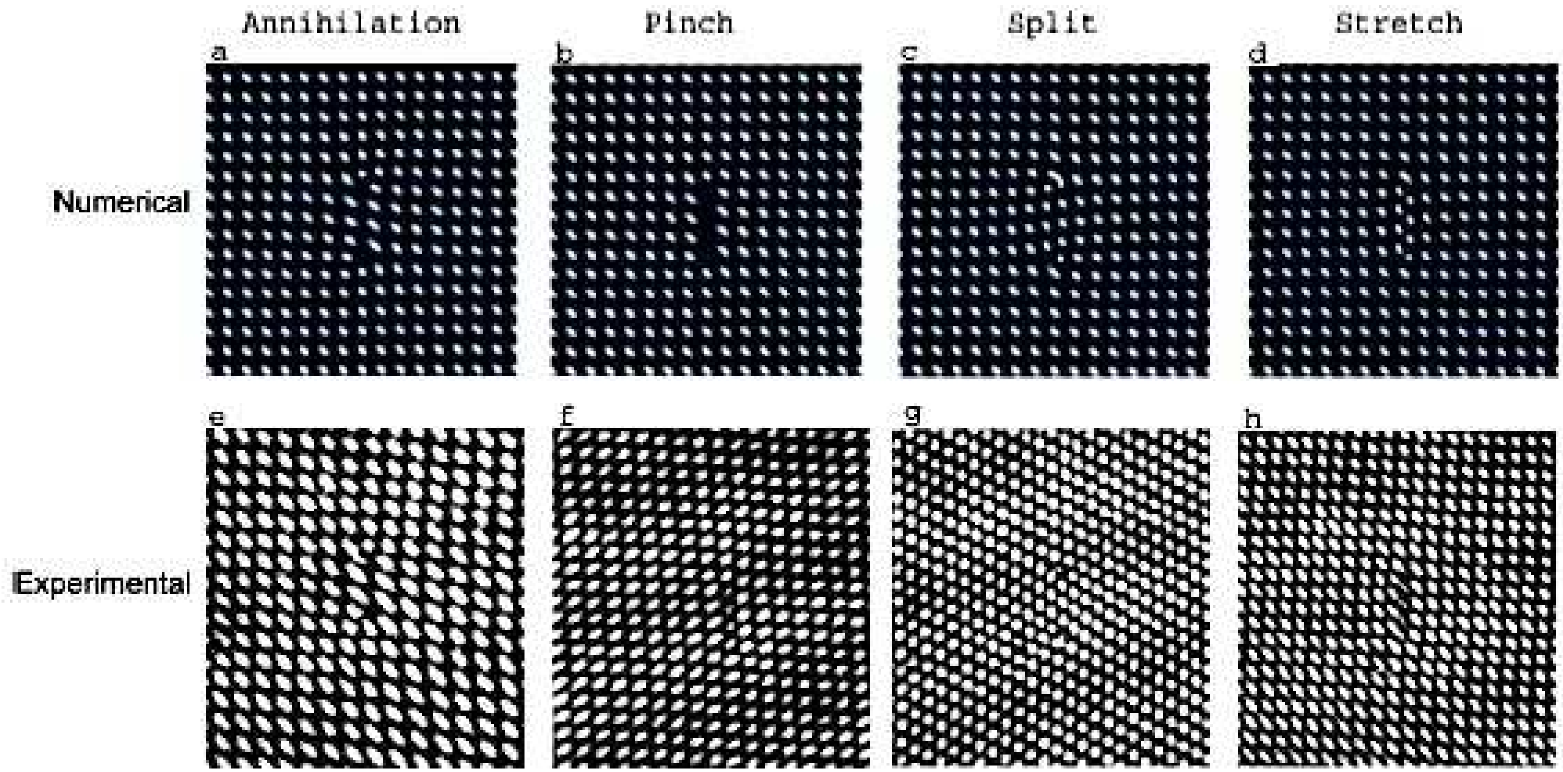
- photonics:
  - optical waveguide arrays (1D)
  - Photonic Crystal Fibers (PCFs)
  - optically-induced in photo-refractive crystals
- Bose-Einstein condensates
- solid state physics
- ...

Irregular lattices



- point defects (e.g., vacancy)
- line defects (e.g., edge-dislocation)
- quasicrystal structures (e.g., Penrose quasicrystal)

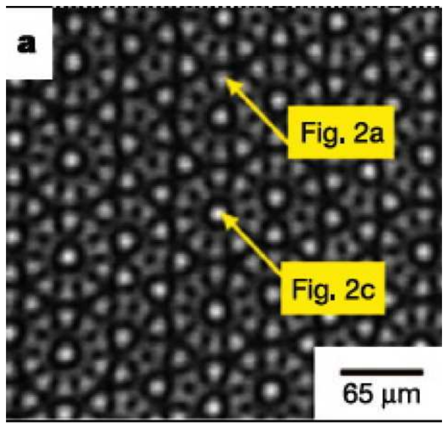
Fabrication of defects

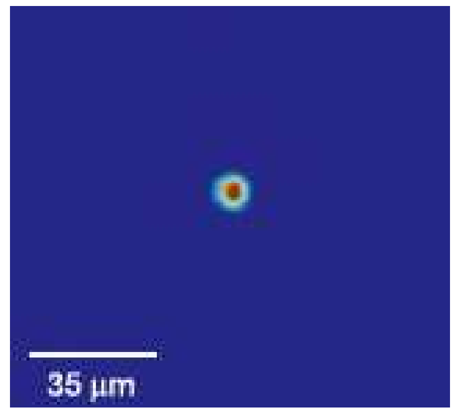


- manipulate defects & dislocations by interference of plane waves and spiral-phase masks ($L \sim$ 0.2mm)
- Schonbrun and Piestun, *Opt. Eng.* '06

Quasi-crystal photonic lattices

Freedman *et al.* ’06: solitons embedded in photonic quasicrystals





Localized nonlinear modes

nonlinear Schrödinger (**NLS**) eq.

$$i\psi_z + \Delta\psi - V(\vec{x})\psi + |\psi|^2\psi = 0$$

look for modes:

$$\psi(\vec{x}, z) = f(\vec{x})e^{-i\mu z} \Longrightarrow [\mu + \Delta - V(\vec{x}) + |f|^2]f = 0$$

$f(\vec{x}) =$ real & *localized*, $P := \iint |f(\vec{x})|^2 \, dxdy < \infty$

- $V(\vec{x}) \equiv 0$ (homogeneous):
  - “Townes soliton” when $\mu < 0$
  - collapse in (2+1)D and higher dimensions
- periodic $V(\vec{x})$:
  - band-gaps; localized modes (lattice solitons)
  - some theory; mostly computational
  - lattice solitons recently observed in experiments

Computation of solitons

$$[\mu + \Delta - V(\vec{x}) + |f|^2]f = 0$$

fixed-point spectral iterations (Ablowitz and Musslimani, 2005)

$$\hat{f}(\vec{k}) = \hat{R}[\hat{f}] \equiv \frac{(r+\mu)\hat{f} + \mathcal{F}\left\{[|f|^2 - V(\vec{x})]f\right\}}{r + |\vec{k}|^2}\ , \quad r > 0$$

renormalize: $f(\vec{x}) = \lambda w(\vec{x})$. Iterate $\hat{w}_{n+1} = \lambda_n^{-1}\hat{R}[\lambda_n \hat{w}_n]$

coupled algebraic condition:

$$\iint_{-\infty}^{+\infty} |\hat{w}_n(\nu)|^2 \, d\nu = \lambda_n^{-1} \iint_{-\infty}^{+\infty} \hat{R}[\lambda_n \hat{w}_n]\hat{w}_n^*(\nu) \, d\nu \ .$$

Renormalization method

$$\begin{cases} \hat{w}_{n+1} & = \lambda_n^{-1} \hat{R}[\lambda_n \hat{w}_n] \,, \quad n = 1, 2, \cdots \\ \iint_{-\infty}^{+\infty} |\hat{w}_n(\nu)|^2 \, d\nu & = \lambda_n^{-1} \iint_{-\infty}^{+\infty} \hat{R}[\lambda_n \hat{w}_n] \hat{w}_n^*(\nu) \, d\nu \,. \end{cases}$$

- initial condition: $w_0(x, y) = e^{-\left[(x-x_0)^2+(y-y_0)^2\right]}$
- soln: $f(\vec{x}) = \lambda w(\vec{x})$
- convergence:

$$\|f_{n+1} - f_n\|_\infty < 10^{-10} \,, \quad \left| \frac{\lambda_{n+1}}{\lambda_n} - 1 \right| < 10^{-10}$$

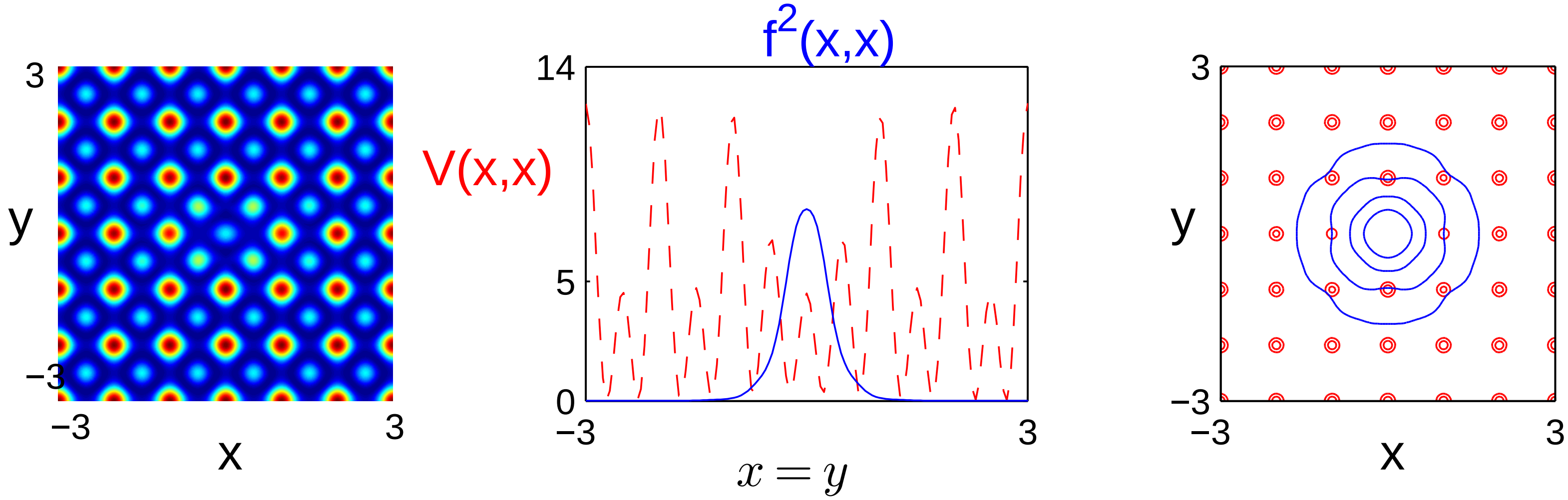
- usually convergence is reached quickly

Vacancy solitons

$$V(x,y) = \frac{V_0}{25}\left|2\cos(Kx) + 2\cos(Ky) + e^{i\theta(x,y)}\right|^2$$

$$\theta(x,y) = \tan^{-1}\left(\frac{y-y_0}{x}\right) - \tan^{-1}\left(\frac{y+y_0}{x}\right), \quad y_0 = \frac{\pi}{K}$$

$$K = 2\pi, V_0 = 12.5, \mu = 0.5$$

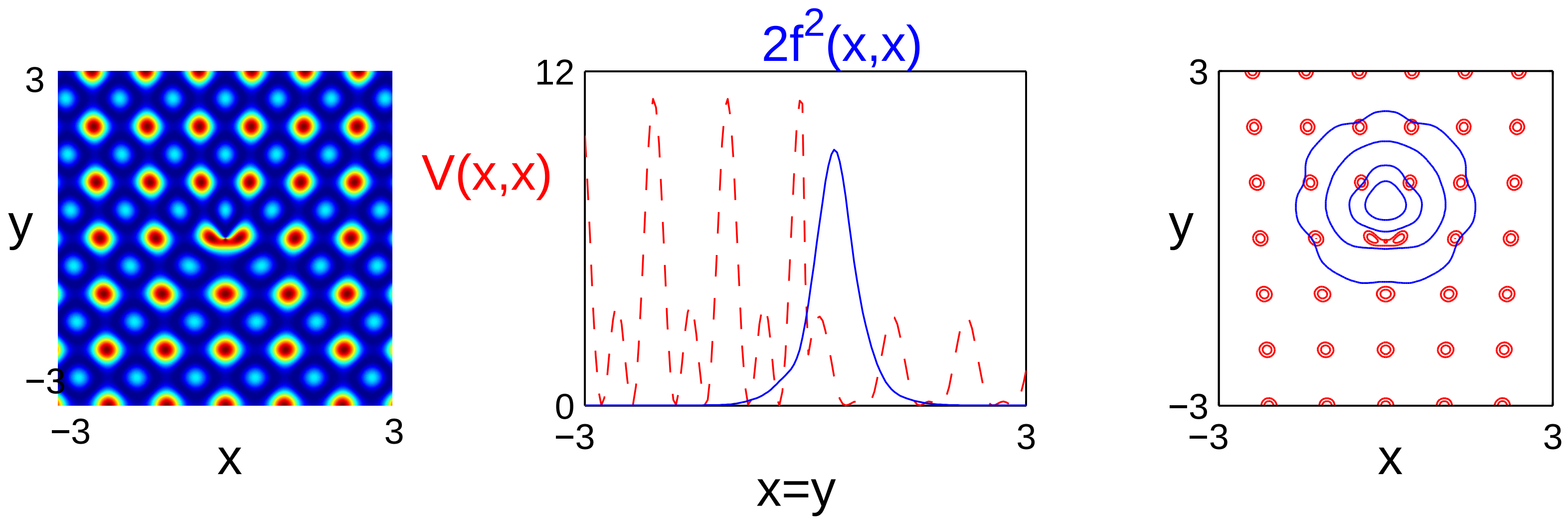


- similar to lattice solitons on minimum of potential

Edge-dislocation solitons

$$V(x,y) = \frac{V_0}{25}\{2\cos[Kx+\theta(x,y)]+2\cos(Ky)+1\}^2$$

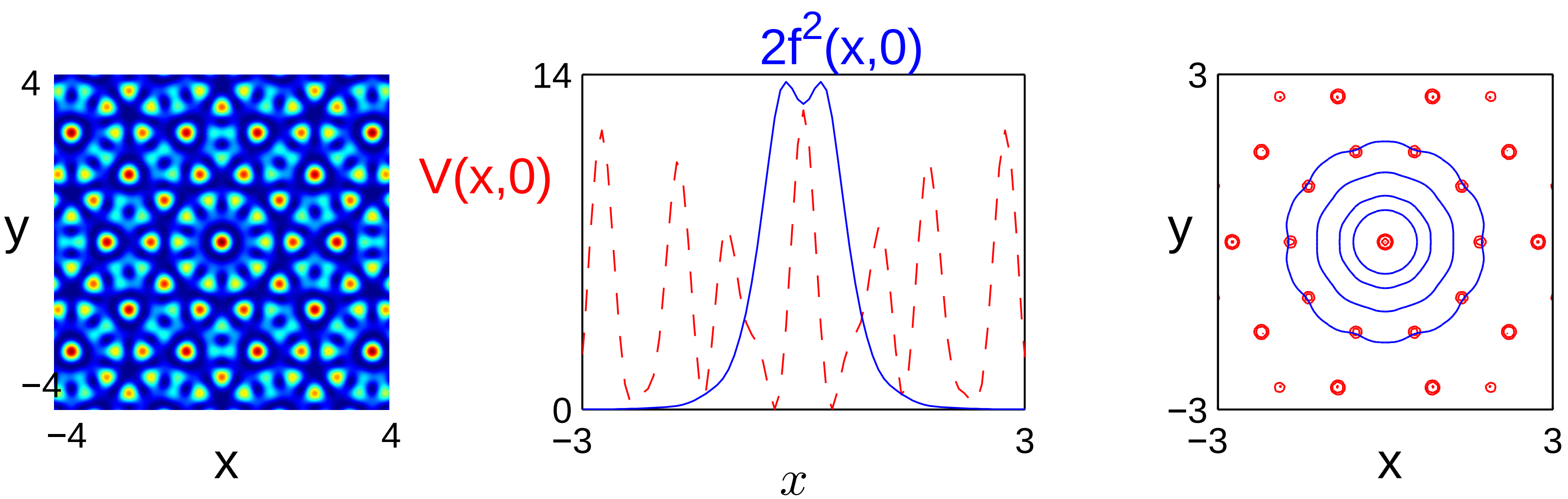
$$\theta(x,y) = \frac{3\pi}{2} - \tan^{-1}\left(\frac{y}{x}\right)$$



- $\mu = 0.5$

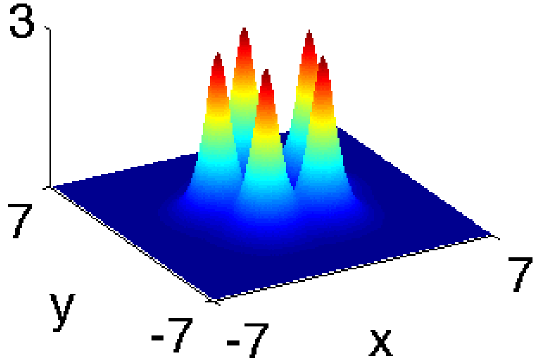
Penrose solitons ($N = 5$)

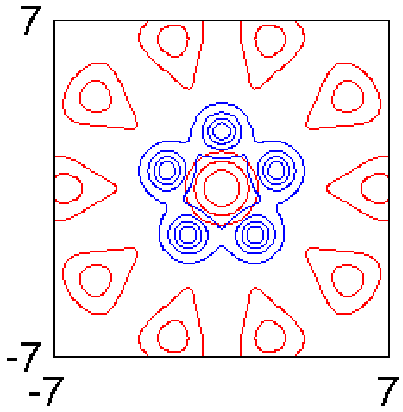
$$V(x,y) = \frac{V_0}{25}\left|\sum_{n=0}^{4} e^{i\vec{k}_n\cdot\vec{x}}\right|^2 \quad , \quad \vec{k}_n = \left(K\cos(\frac{4}{5}\pi n), K\sin(\frac{4}{5}2\pi n)\right)$$

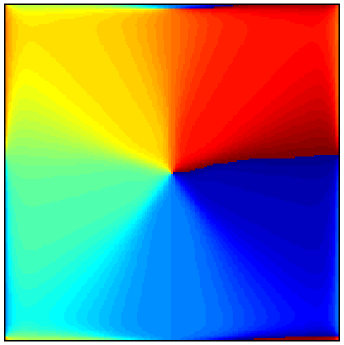


- $\mu = 0.5$
- on lattice maxima wide solitons have a dimple

Vortex quasicrystal solitons







- $N = 5$ and $\mu = -2$
- with Ablowitz, Antar, Bakırtaş, Ilan (in progress)

Dynamics and instabilities

Wave collapse

singularity formation $\|\psi(x,y,z)\|_{H_1} \overset{z\to Z_c}{\longrightarrow} \infty$

in practice singularity occurs at a point: $\max_{x,y}|\psi| \overset{z\to Z_c}{\longrightarrow} \infty$

- strictly nonlinear phenomenon
- can only occur in (2+1)D and higher dimensions
- necessary condition for collapse (Weinstein ’85; Pacciani and Konotop ’06): $P := \iint |\psi_0(x,y)|^2 \geq P_c^{\mathrm{NLS}} \approx 11.7$
- sufficient condition: negative Hamiltonian; generically not-sharp (e.g., Ablowitz, Bakırtaş, Ilan, Physica D ’05)
- conditions apply to any initial conditions

Self-focusing instability

$$i\psi_z + \Delta\psi - V(\vec{x})\psi + |\psi|^2\psi = 0$$

peak amplitude can significantly increase during evolution

self-focusing instability of solitons:

- $\psi(\vec{x}, z) = f(\vec{x})e^{-i\mu z}$, $P := \iint |f(\vec{x})|^2$
- Vakhitov-Kolokolov (**VK**) criterion: need $\frac{dP}{d\mu} < 0$ for stability
- many studies rely on this criterion, but:
  1. only necessary for stability, not sufficient
  2. only linear stability, what about collapse?
  3. only for modes, not general initial conditions

Stability theory

Let $u(x,y) > 0$ be soliton)solution. Then the soliton (+ small perturbation) remains "orbitally stable" during propagation $\leftrightarrow$ **both** *of the following conditions apply:*

1. *slope/power/Vakhitov-Kolokolov condition:* $\frac{\partial P(\mu)}{\partial \mu} < 0$
2. *spectral condition:* $L_+^{(V)} = -\Delta - \mu - 3u^2(x,y) + V(x,y)$ *has exactly one negative eigenvalue [*$\lambda_{1,2}^{(V)} \geq 0$*]*

- Weinstein ('85): proof for pure NLS, i.e, $V(x,y) \equiv 0$
- Floer & Weinstein; Rose & Weinstein; Stuart; Spradlin: extension of proof to certain potentials; with Weinstein (in progress): general potentials including irregular lattices
- spectral condition lesser known and studied

Criteria for instabilities

1. slope and spectral conditions are “equally” important in theorem, but correspond to different instability mechanisms
2. $\frac{dP}{d\mu} \geq 0 \Longrightarrow$ diffraction/self-focusing
3. $\lambda_{1,2}^{(V)} < 0 \Longrightarrow$ soliton drifts across the lattice
4. “strength” depends on size of $\frac{dP}{d\mu}$ and $\lambda_{1,2}^{(V)}$

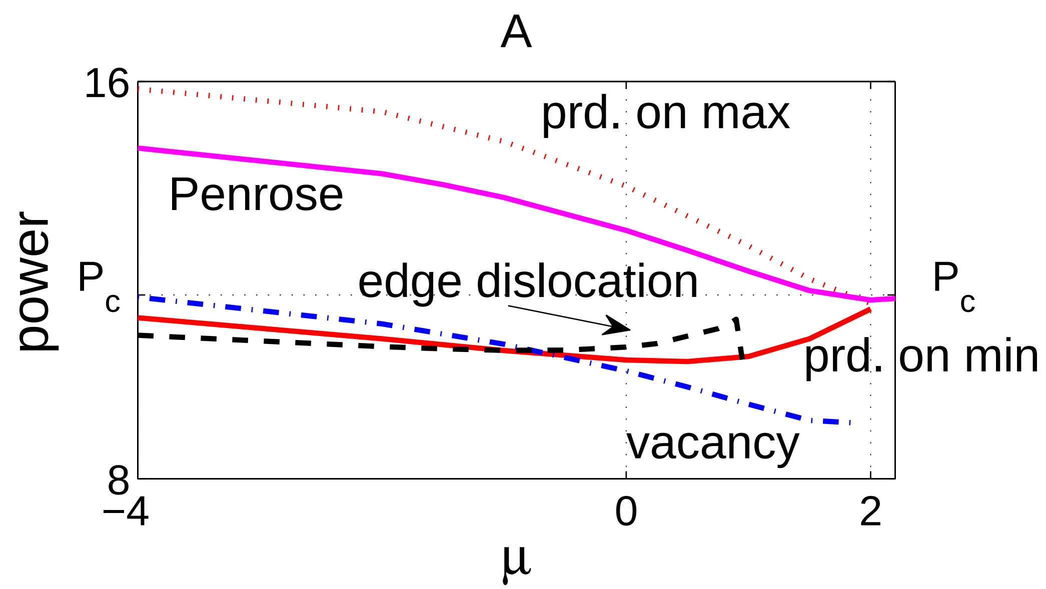
a $\lambda_{1,2}^{(V)} < 0 \ \leftrightarrow$ soliton not on a min

b slope & spectral conditions depend location & width

c collapse when $P > P_c^{(V)} \simeq P_c^{\mathrm{NLS}} \approx 11.7$

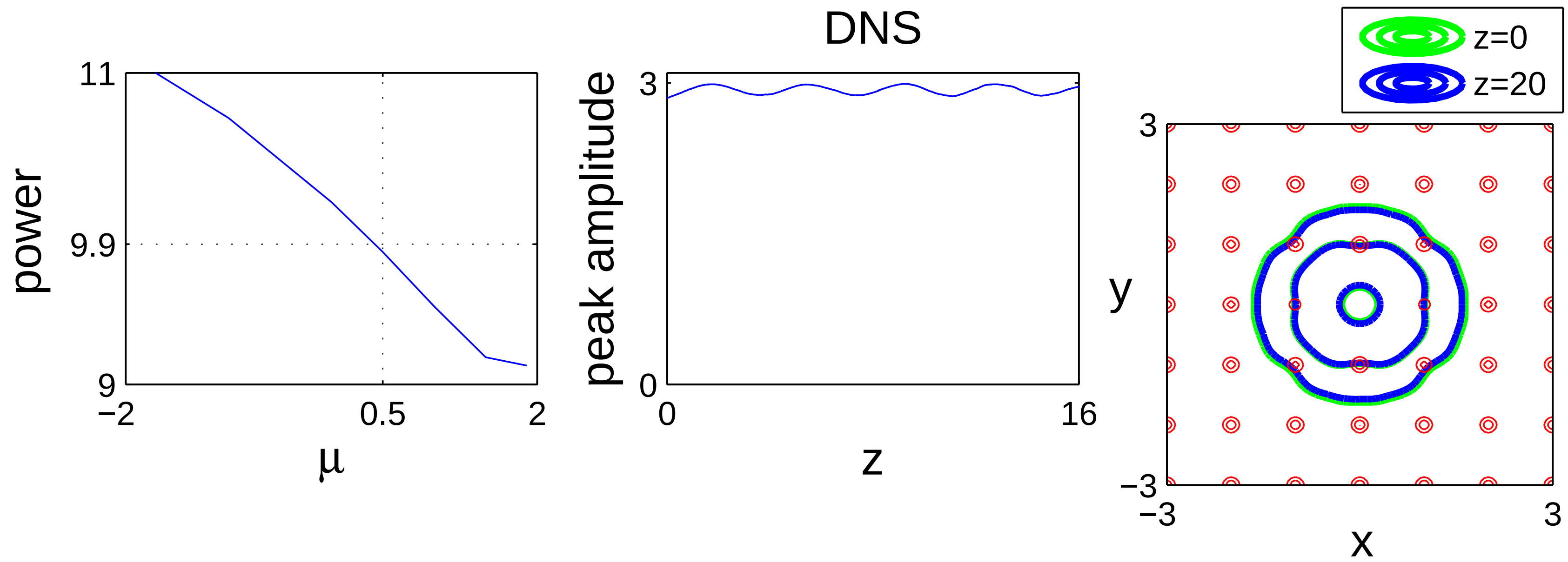
with Ablowitz, Fibich, Sivan and Weinstein (in progress)

Power condition



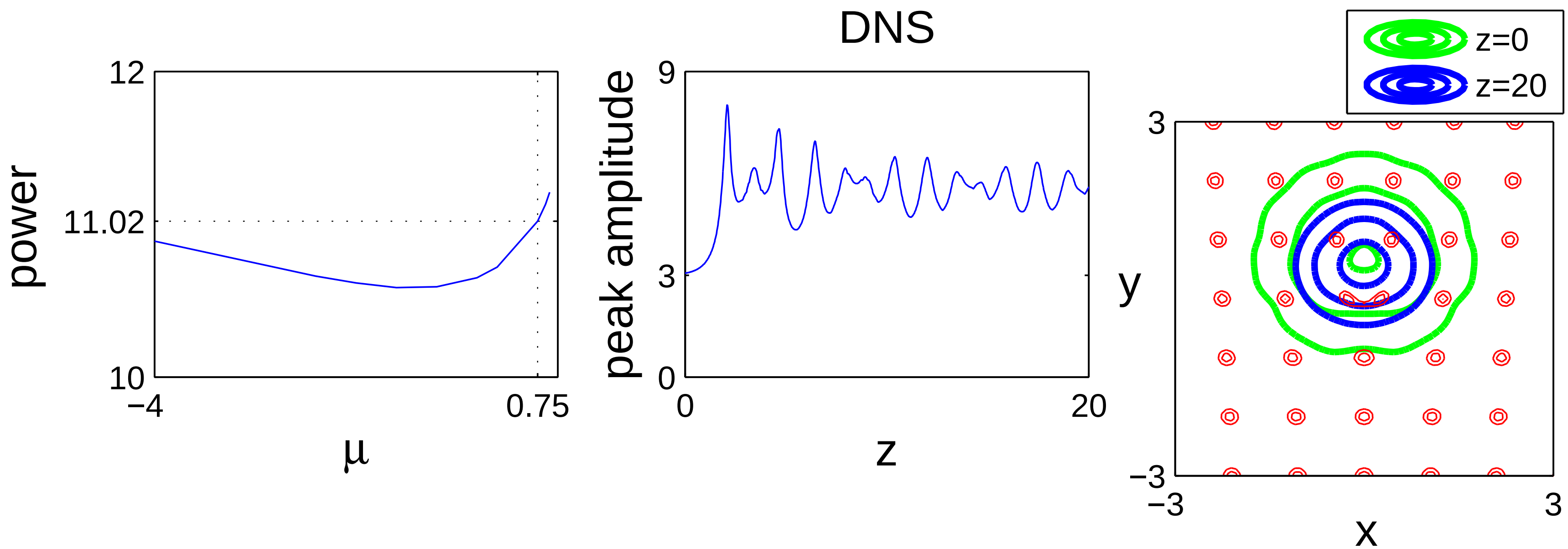
- $V_0 = 12.5,\ K = 2\pi$
- VK: need $\frac{dP}{d\mu} < 0$ for stability
- based on VK: solitons is likely to be stable when sufficiently far from bandgap edge (not too wide)

Evolution of vacancy solitons



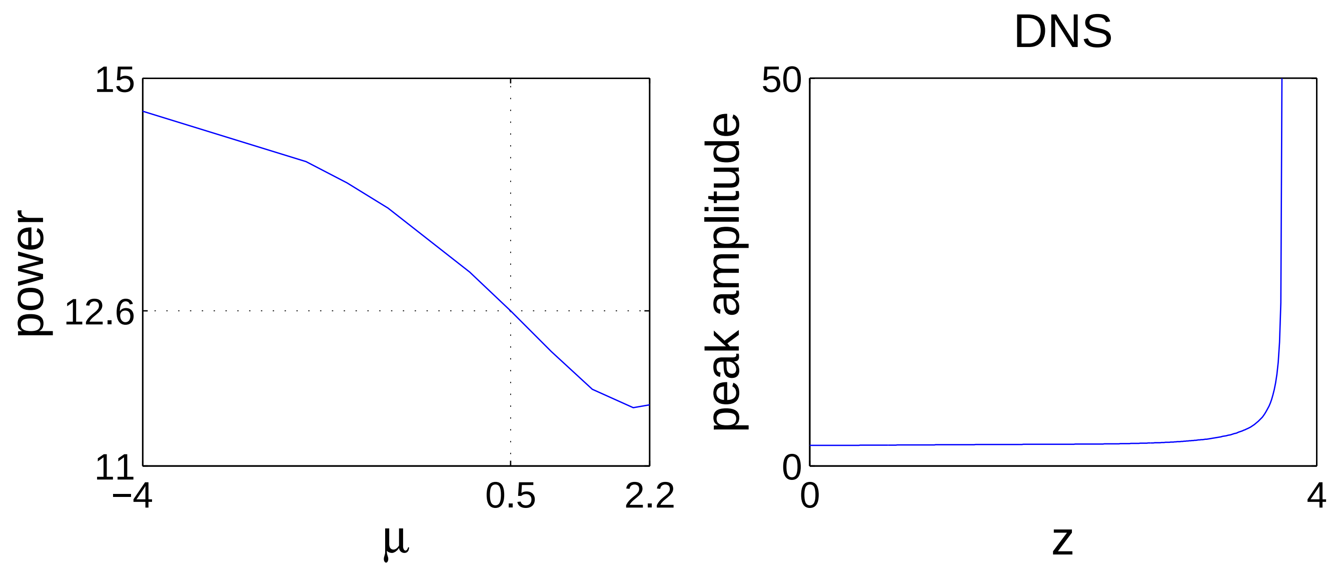
- $\mu = 0.5 \Rightarrow \frac{dP}{d\mu} < 0 \longrightarrow$ stable
- Direct Numerical Simulations (DNS) of (2+1)D NLS
  Initial conditions: mode + 1% noise
- DNS: small focusing-defocusing oscillations

Evolution of edge dislocation solitons



- $\mu = 0.75 \Rightarrow \frac{dP}{d\mu} > 0 \longrightarrow$ linearly unstable
- DNS: focusing-defocusing oscillations

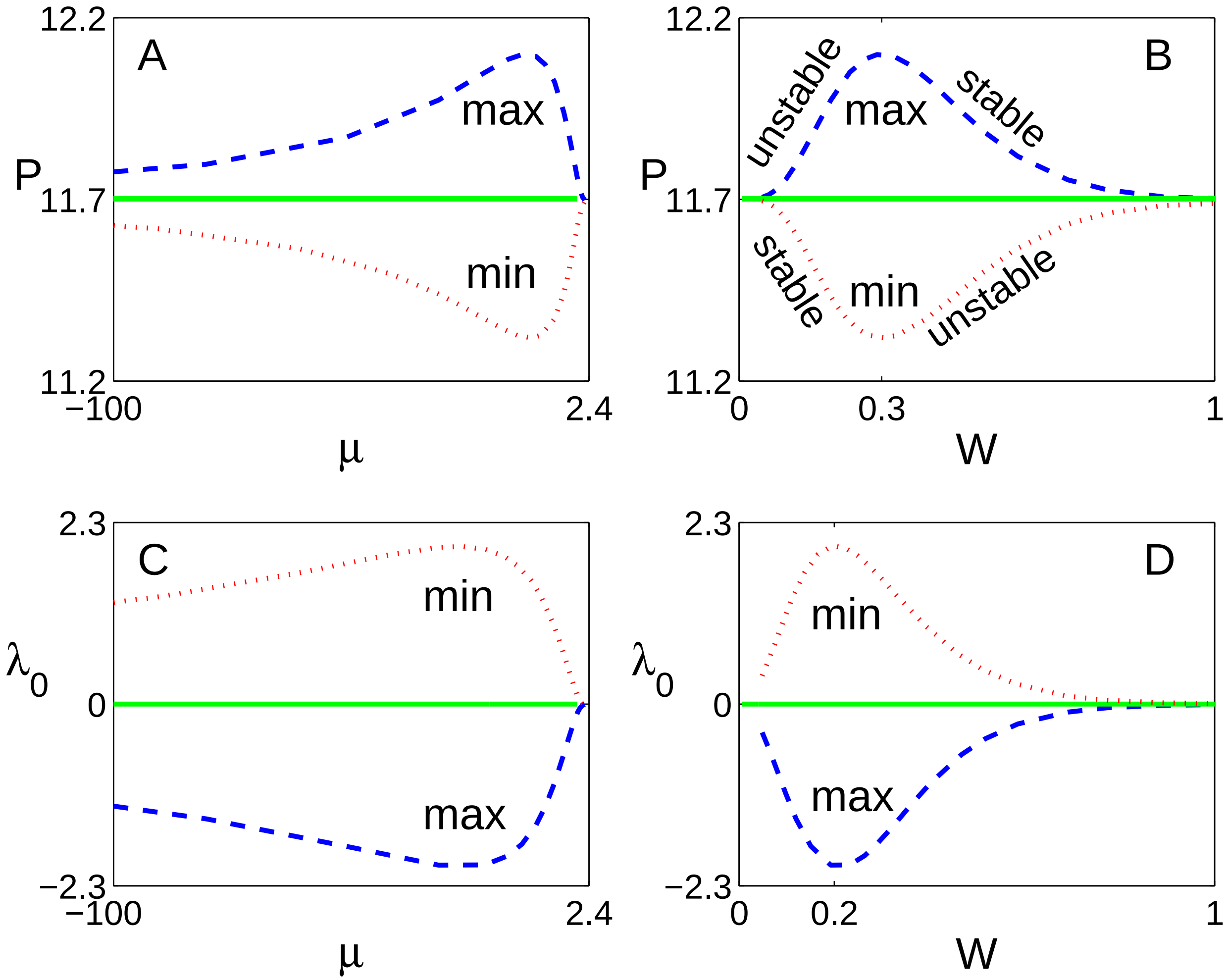
Evolution of Penrose solitons



- $\mu = 0.5 \Longrightarrow \frac{dP}{d\mu} < 0 \longrightarrow$ linearly stable?
- DNS: collapse.
- power condition is only necessary, not sufficient

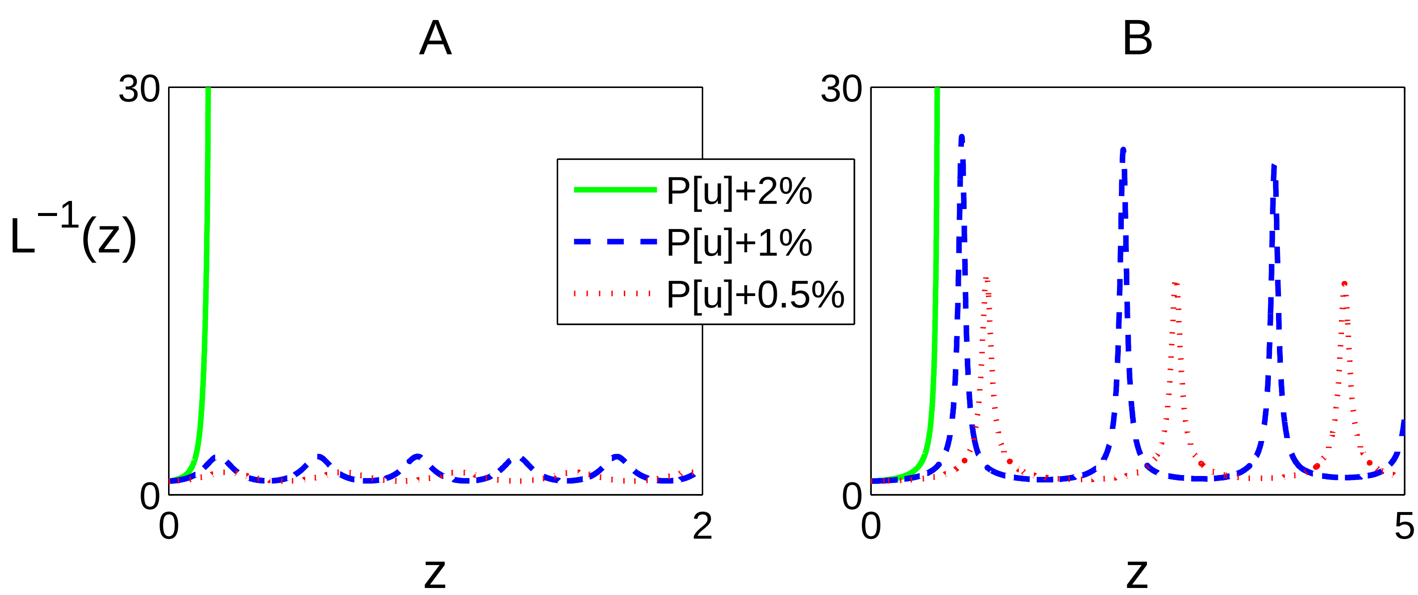
Power and spectral conditions

periodic square lattice $V(x,y) = 2.5\left[\cos^2(2\pi x) + \cos^2(2\pi y)\right]$



Self-focusing and collapse

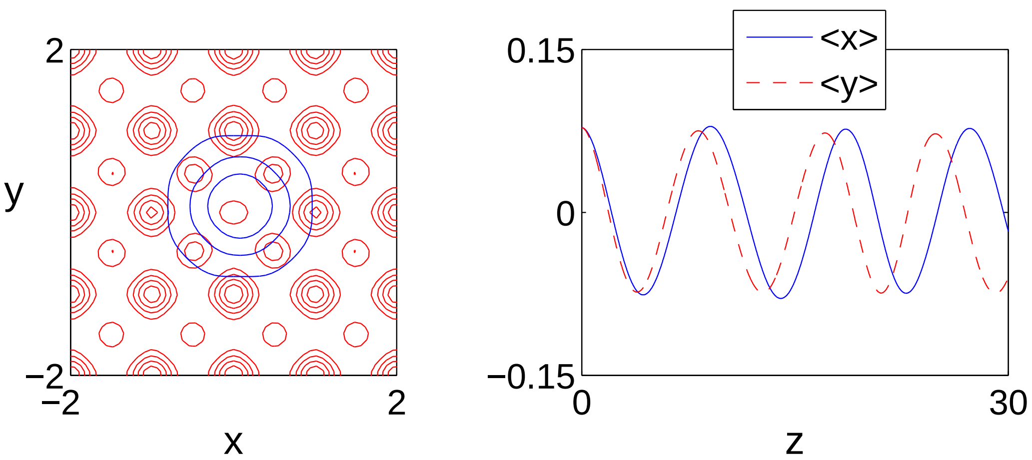
- Use power-perturbed mode as initial conditions
- $\mu = -31$ (left) and $\mu = -3$ (right) have $P[u] \approx 0.98 P_c^{\rm NLS}$, but opposite sign of $\frac{dP}{d\mu}$
- $L^{-1}(z) := \max_{(x,y)} |\psi(x,y,z)|^2 / \max)(x,y)|\psi_0(x,y)|^2$



Drift instability

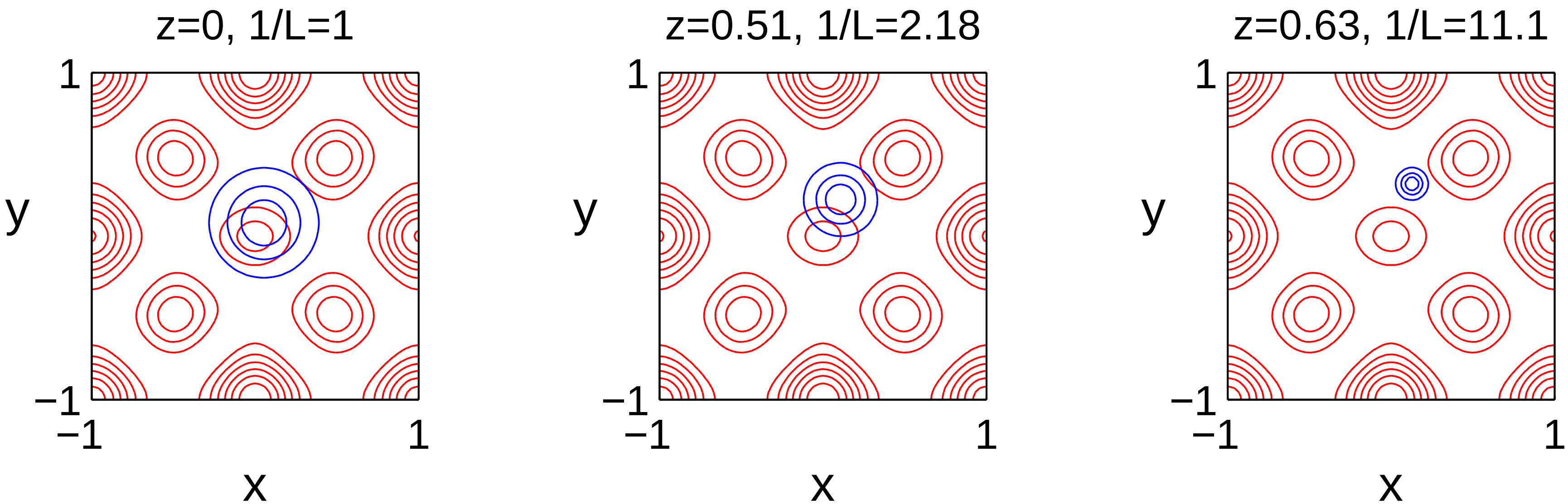
- drift (lateral dislocation) observed during evolution,
- associated with violation of spectral condition $\lambda_{1,2}^{(V)} < 0$
- occurs generically when initial conditions are on potential max
- studied by Pelinovsky in 1D lattices
- Fibich and Sivan: nonlinear lattices; further investigations in progress

Drift during evolution



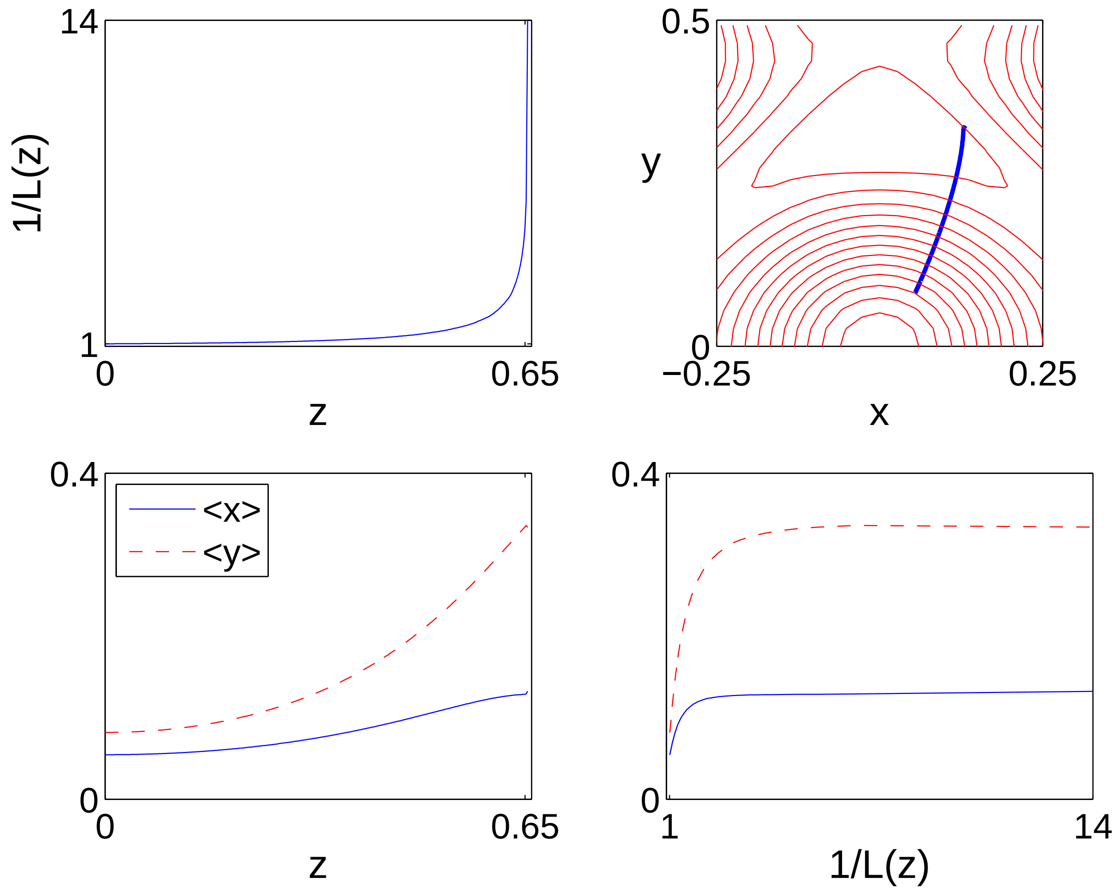
- drift + oscillations

Drift – cont.



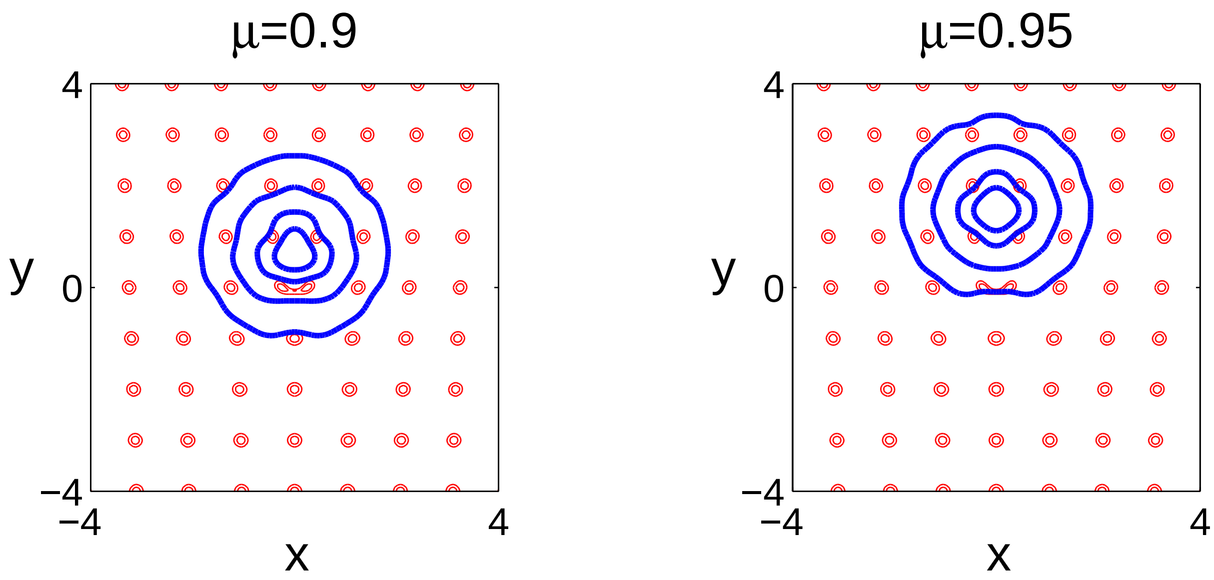
- drift + collapse

Drift – cont.



- drift continues during collapse process

Drift during mode computation



- drift can also be observed during iterations of Spectral Renormalization method

Summary

- 2D solitons in periodic lattices well known and observed
- theoretical/computational studies of 2D solitons in irregular lattices: vacancy defects, edge-dislocations, and quasicrystal structures, extensions to vortex and Bessel solitons
- rigorous theory, asymptotics and systematic computations give insight into soliton instabilities

Thank you for your attention!