Analysing quality with generalized kinetic methods

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Idea

Generalized kinetic models represent a fruitful predictive and descriptive tool in the area of the social sciences.

These models transfer the methodology developed for systems of a great number of interacting particles (typically in the field of kinetic theory of classical particles) to various other fields of research.

Generalized Boltzmann models can be applied to perform the difficult task of treating advanced and complex systems such as those that refer to human individuals.

Overview

A statistical picture ("Boltzmann model") is developed to describe the time evolution of a macroscopic variable related to the quality of a fully developed system such as a medical service.

This is done by means of a microscopic state variable, which in the literature is denoted by activity of the actors, driven by actions both of internal and of external nature.

The aim is to propose a convenient set of evolution equations that may describe the dynamics of the probability density functions about this variable.

The analysis aims at singling out and characterizing the physical variables that control the long times (or average) dynamics, and at predicting the effect that possible readjustments of the structure may produce on it.

Description of the system

- The structure has individuals as targets of its efforts.
  [E.g.: patients, customers, users, ...]
  These constitute the first populations of actors.
- The structure is run by individuals that provide a service to the former actors.
  [E.g.: medical staff, clerks, agents, ...]
  These constitute a second population of actors.
- The service is subject to a dynamics that runs on different time scales, often related to the different populations of actors.
  [E.g.: first aid to patients on a fast time scale, turnover shifts on a median time scale, staff reorganizations on a long time scale.]
- Actors are identified by their population and by a unique state variable: the activity. Kinematic variables of the actors have no relevance on the dynamics.

State variable: activity

If referred to actors of different populations the activity has different meanings.
For instance, in the case of a patient the quality is related to the satisfaction he feels about the service he receives, for an agent it is related to the stress he is subject to when performing his duty, for a piece of hardware it simply depends on its efficiency and suitability. More specifically:

- u_1 addresses the psychotic behavior of the patients;
- u_2 addresses the stress of the operators;
- u_3 addresses the efficiency of the hardware.

With suitable parametrizations it is possible to code the values of the different state variables on the same interval, for instance $u_i \in I_i = [0,1]$ for $i = 1,2,3$, with $0 \equiv$ “good” and $1 \equiv$ “bad” .

Color code

- green: everything is fine in the ward and there are no dominant emotions in the staff;
- yellow: in the unit there might be one or more patients whose psychotic behavior makes the operators feel tired and disquiet;
- orange: the unit might be crowded, and there might be patients in a critical state whose psychotic a/o slightly violent behavior makes the operators feel worried, anxious;
- red: the unit is very crowded and one or more patients in a critical state behave violently, thus putting in a pre-alarm state most operators: feelings of irritation and fear spread out;
- fiery red: the situation is similar to the previous one, and in addition it becomes necessary to call for external help, like 911, or to tie up one or more patients.

The activity is assumed to be a scalar random variable, one per each of the system populations $u_i \in I_i = [0,1]$, defined by the processes

$$u_i : t \in [0,T] \rightarrow u_i(t) \in I_i \equiv [0,1]$$

The corresponding probability density functions

$$f_i : (t,u) \in [0,T] \times I_i \rightarrow f_i(t,u) \in [0,\infty[,$$

describing how the individuals of each population are distributed with respect to the state variable, are the objects of our study.
In particular, we aim at formulating an initial value problem: given the probability density functions at time $t = 0$ (which will be assumed to be truncated normal distributions with respect to the state variable u, with assigned mean and variance), find the equations that give their time evolution.

Assumption 1.

The system is composed by two populations P_1, P_2 of actors. Population P_1 is composed by $N_1 = N_1(t)$ individuals, the users of the service; population P_2 by $N_2 = N_2(t)$ individuals, the service agents.

[I'll ignore the third population for the sake of simplicity.]

Mass functions $N_i : t \in [0, T] \to N_i(t)$ are assumed to be stepwise constant, integer valued, right continuous, and known as data of the problem.

Discontinuities may be found only at the endpoints $t_m = m\tau$, $m = 0, 1, \ldots, M \in \mathbb{N}$, of the evolution intervals.

Assumption 2.

Individuals of the same population are identical and only addressed to by the state (random) variable denoting their activity. The system description is obtained by the set of density functions over the individuals states.

The probability density functions

$$f_i : (t,\cdot) \in I_i \subset \mathbb{R} \longrightarrow f_i(t,\cdot) \in [0,\infty)$$

are such that the event $u_i \in [c_1, c_2] \subset I_i$, which refers about the outcome of the cited (random, uncertain) activity measurement at time $t \in [0,T]$, has probability given by

$$P(t; c_1 \leq u_i \leq c_2) = \int_{c_1}^{c_2} f_i(t,x)dx$$

Functions f_i are sufficiently regular; in particular, they are $C^2(I_i)$, piecewise continuous with respect to t and normalized to 1

$$\int_0^1 f_i(t,x)dx = 1 \qquad \forall t \in [0,T]$$

The **mean activity of population** $\mathbf{P}_i$

$$U_i(t) = N_i(t) \int_0^1 x f_i(t,x) dx \qquad \forall t \in [0,T]$$

plays the role of the macroscopic, “measurable” quantity.
It might be reasonable to weight differently the mean activity of each population in evaluating the global atmosphere of the system.
On these account, we assume that real normalizing constants $\alpha_1, \alpha_2 \in \mathbb{R}$ may be properly defined such that

$$U(t) = \alpha_1 U_1(t) + \alpha_2 U_2(t)$$

correctly refers about the global atmosphere (expected quality) of the service at time t.

Assumption 3.

Each individual is subject to actions of external and of internal nature.

Actions of internal nature (interactions) are identified, as usual in kinetic theories, by means of convenient encounter frequencies and change of state probabilities.

Actions of external nature act by means of a term with the structure of a field.

Only instantaneous interactions are considered.

Evolution equations are meant to describe the dynamics of the probability density functions f_i that refer about the status of actors of population P_i at time t [more precisely, $N_i(t)\, f_i(t,x)$ denotes the expected fraction of the total number $N_i(t)$ of actors of population P_i that are in status x].

Mass balance equations

Assumption 4.
The total variation rate of each density function f_i, namely the sum of the direct variation with respect to time plus a flow term due to external actions and (possibly) to internal actions of global character, is equal to the balance between a “gain”term and a “loss” term referred to the specified density and due to internal “interactions”.
In formula:

$$\partial_t f_i + \partial_x \Phi_i = G_i[\mathbf{f}] - L_i[\mathbf{f}] \qquad i = 1, 2$$

No change of populations are (obviously) allowed.
Input or output of actors are taken into account only at the instants $t_m = m\tau, m = 0, 1, 2, ..., M$, when the evolution undergoes its global periodic variations.

Mean field term

As the flux terms are concerned, we borrow our reasoning from the theory of a (hypothetical) flow.

Assumption 5.

The convective term on probability distribution functions $f_i(t,x)$ has the structure of a (local) net flow: $\partial_u \Phi_i$. The flow Φ_i may be identified by

$$\Phi_i(t,x) = K_i[\mathbf{f}](t,x) f_i(t,x) + c_i(t,x)\partial_x f_i(t,x), \qquad x \in I_i$$

The coefficient $K_i[\mathbf{f}]$, to be thought of as a drift velocity, represents the internally induced speed of change due to actions of global character that may be ascribed to ensembles of actors of the various populations.

The system is indirectly driven by a set of events that happen non-uniformly and unexpectedly (extra-events).
Each extra-event is identified by a real value e_q

$$\mathbb{E} = \{e_1, e_2, \ldots, e_E\}$$

that collects all the possible chances, both for "good" events and for "bad" events.
The effects of an extra-event start at the beginning of the evolution subinterval $[t_m, t_{m+1}[$ and last for that interval only.
The final action of all extra-events upon the system is of a collective nature, and is identified by the "macroscopic" function

$$E(t) = \sum_{q=1}^{E} e_q \delta_q^{(m)}, \qquad t \in [t_m, t_{m+1}[$$

where $\delta_q^{(m)} = 1$ if the q-th event happened at time t_m and 0 otherwise.

Apart from a suitable normalization function necessary to annihilate the flux Φ_i at the endpoints of the interval $[0,1]$, the drift K_i is given by

$$K_i[\mathbf{f}](t,u) = \chi(u)\left[\beta_i U(t) + \gamma_i E(t)\right]$$

with $\chi(0) = \chi(1) \equiv 0$.
The flux also contains a diffusive term, thought to be due to a global dynamics induced on the actors and on their states by actions of external origin, which is proportional to the gradient of the probability density functions.

Interactions

Assumption 6.
Interactions modify the state of a test actor, not its population, with a rate specified by convenient functions: $\eta_i, \eta_{i,j}$, that refer about the frequencies of his relations with field actors. State changes are stochastic events, specified by convenient transition (conditional) probability density functions $\psi_i, \psi_{i,j}$.

The following sets of (regular) functions are assumed to exist. In all of them, $t \in [0,T]$, $i, j = 1, 2$. Each of them may depend on time not only as an external parameter, but also because their dependence on expectations over the densities $\mathbf{f} = [f_1, f_2]$. This kind of implicit dependence on the densities $\mathbf{f}$ is recalled by the square bracket notation.

- $\eta_i(t,x)$ rate of events wherein an individual of population P_i autonomously reflects about modifying his state x.
- $\eta_{i,j}(t,x,y)$ rate of events wherein an individual of population P_i in the state $x \in I_i$ "encounters" an individual of population P_j in the state $y \in I_j$.
- $\psi_i[\mathbf{f}](t,x;x')$ probability density function about the outgoing state $x' \in I_i$ of a test individual of population P_i in the state $x \in I_i$ after an event wherein he autonomously reflects about the possibility of modifying his state x.
- $\psi_{i,j}[\mathbf{f}](t,x,y;x')$ probability density function about the outgoing state $x' \in I_i$ of a test individual of population P_i in the state $x \in I_i$ after an event wherein he encounters an individual of population P_j in the state $y \in I_j$.

The nonlinear evolution equations for the densities f_1 are given by the statistical balances

$$\frac{\partial f_i}{\partial t}(t,u)+\frac{\partial}{\partial u}\left(K_i[\mathbf{f}]\,f_i+c_i\partial_u f_i\right)(u,t)=G_i[\mathbf{f}]\,(t,u)-L_i[\mathbf{f}]\,(t,u)$$

where

$$\begin{aligned}G_i[\mathbf{f}]\,(t,u) &= \int_0^1 \eta_i(t,x)\psi_i[\mathbf{f}](t,x;u)\,f_i(t,x)dx \\ &\quad + \sum_{j=1}^{2}\int_0^1\int_0^1 \eta_{i,j}(t,x,y)\psi_{i,j}[\mathbf{f}](t,x,y;u)\,f_i(t,x)\,f_j(t,y)dxdy\end{aligned}$$

$$L_i[\mathbf{f}]\,(t,u) = f_i(t,u)\eta_i(t,u)+f_i(t,u)\sum_{j=1}^{2}\int_0^1 \psi_{i,j}(t,u,y)\,f_j(t,y)dy$$

and

$$K_i[\mathbf{f}](t,u)=\chi(u)\left[\beta_i U(t)+\gamma_i E(t)\right]$$

Assumption 7.

Probabilities are computed as expectations of suitable probability density functions ψ, stated in Assumption 6, that may be selected among those that are a priori identified by their most probable value μ and their uncertainty σ.
For instance, it has already proved to be convenient a normal distribution truncated on the set $I = [0,1]$, with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$:

$$\psi(u;\mu,\sigma) = \frac{\exp\left[-(u-\mu)^2/2\sigma^2\right]}{\int_0^1 \exp\left[-(u-\mu)^2/2\sigma^2\right] du}$$

In the simplest possible model, the frequency of interaction is uniform, i.e., it is constant in time and independent of the state of the interacting individuals:

$$\eta_{i,j}(v,w) = \eta_{ij} \geq 0 \qquad i,j = 1,2$$

For instance,

$$\eta_{2,1}(v,w) = \eta_{1,2}(v,w) = \eta_0 > 0$$

and $\eta_{2,2}(v,w) = \varepsilon\eta_0$ with $0 < \varepsilon < 1$.
[reasonable numbers would correspond to interactions taking place 10 and 3 times a day, respectively]
The interaction frequency between two operators

$$\eta_{2,2}(v,w) = \eta_1$$

and η_1 should correspond to 15 interactions per day.

In a more sophisticated model, one can assume that the interactions, especially between two patients, are **short range** interactions, in the sense that interactions between two patients whose states are very different from one another are highly unlikely or even non-existent.
As far as the interactions between a patient and an operator are concerned, we may assume that the frequency depends on the state of the patient, but it does not depend too much on the operator state (the operator is a professional; however, when the state of the operator is critical or sub-critical we may expect the encounter rate to decrease, if nothing else for lack of time).

The transition function is a probability density with respect to the first variable u and therefore

$$\int_0^1 \psi_{ij}(u; v, w)\ du = 1.$$

These functions are chosen among probability densities that are completely characterized once the mean value m and the variance σ of the stochastic variable u are assigned. We assume that the mean value m of the state after the interaction (for the interacting individual originally in the state v) and the variance σ are given a priori for each value of i, j and w and that this is sufficient to assign the value of the function ψ for all u.

m and σ are then known functions of the state variables of the interacting individuals for each type of interaction procedure:

$$m = m_{ij}(v,w) \qquad \sigma = \sigma_{ij}(v,w) > 0$$

and

$$\psi_{ij}(u;v,w) = \psi(u;m_{ij}(v,w), \sigma_{ij}(v,w))$$

with

$$m_{ij}(v,w) = \int_0^1 u\,\psi_{ij}(u;v,w)\,du$$

$$\sigma_{ij}(v,w) = \int_0^1 (u - m_{ij}(v,w))^2 \psi_{ij}(u;v,w)\,du$$

specifying that m_{ij} and σ_{ij} are, respectively, the mean value and the standard deviation of the distribution ψ_{ij}.

Assume the variance to be constant, independent of the state variables.
Possible types of interactions are:

(i) interaction of altruistic nature, giving for both the interacting individuals a mean value m inside the interval (v,w) of the initial states; for instance, m can be the average

$$m(v,w) = (v+w)/2$$

(ii) interaction of competitive (or egoistic) nature, giving for the individual initially in the state v a mean value m which is larger than v if v is larger than w, and smaller than v otherwise;

(iii) one could assume that, at least on a short time scale, the interaction is beneficial to both interacting individuals, and therefore such that $m(v,w) \geq \max(v,w)$.

(iv) ...

The interactions patient-operator are of a different nature. One would expect the state variable of the interacting operator to be very modestly affected (or not affected at all) by the state of the patient, unless the patient is a critical or sub-critical state.
As far as the effect of the interaction on the patient is concerned, we expect it to depend mostly on the state of the patient, and only slightly on the state of the operator. Moreover, we assume that the mean value should be always larger or equal to the initial value v, unless the operator is in a critical or sub-critical state, and the empirical datum suggests that the 'benefit' of the interaction with an operator is larger the smaller is v.